$X^{d=\text{even}}$ Compact complex mfd. Herm. metric g on TX

$E \to X$ holomorphic vector bundle, Hermitian metric h

$A^{p,q}(E) = \{$ sm. E-valued (p,q)-form $\}$.

$$\phi = \sum_{a,I,J} \phi^a_{IJ} \, s_a \otimes \underbrace{dz_I}_{= dz_{i_1} \wedge \cdots \wedge dz_{i_p}} \wedge \underbrace{d\bar{z}_J}_{= d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}}.$$

$$(\phi, \psi)_{L^2} = \int_X \langle \phi, \psi \rangle \, dv_g$$

$$= \int_X \phi \wedge \bar{*}_E \psi$$

$$= \int_X (\underset{(p,q)}{\phi}, \underset{(n-q,n-p)}{*\psi})_h$$

$$= \int_X (\phi \wedge \overline{*\psi})_h$$

r-forms: $\underbrace{\langle \eta_1, \eta_2 \rangle dv_g}_{d} = \underset{r}{\eta_1} \wedge \underset{d-r}{\overline{*\eta_2}}$

$*: \Lambda^{p,q} \to \Lambda^{n-q,n-p}$

$\phi = \eta_1 \otimes s_1$, $(\phi, *\psi) = \eta_1 \wedge \bar{\eta}_2 (s_1, s_2)_h$

$*\psi = \eta_2 \otimes s_2$

$$\bar{*}_E : \Lambda^{p,q}(E) \to \Lambda^{n-p,n-q} E^*$$

$$\eta \otimes s \mapsto \overline{*\eta} \otimes \tau(s)$$

conjugate complex linear

$\tau : E \to E^*$

$s \mapsto \underbrace{\langle \cdot, s \rangle_h}$.

$h : E \times \bar{E} \to \mathbb{C}$

$(s_1, s_2) \mapsto (s_1, s_2)_h$.

$$\bar{*}_E \bar{*}_E = **\eta \otimes s = (-1)^r \cdot \eta \otimes s$$

$$\bar{\partial}(\eta \otimes s) = \bar{\partial}\eta \otimes s + (-1)^{p+q} \eta \otimes \bar{\partial} s$$

$$(\bar{\partial}\phi, \psi)_{L^2} = (\phi, \bar{\partial}^* \psi)_{L^2}.$$

$\bar{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$

$\bar{\partial}^* : A^{p,q+1}(E) \to A^{p,q}(E)$.

$$(\bar\partial\phi,\psi)_{L^2} = \int_X (\bar\partial\phi\wedge *\bar\psi)_h = \int_X \underbrace{\bar\partial\phi\wedge\bar{*}_E\psi} \qquad p+q=r$$

$$= (-1)^{r+1}\int\phi\wedge\bar\partial\bar{*}_E\psi \qquad d(\phi\wedge\bar{*}_E\psi) - \phi\wedge\bar\partial\bar{*}_E\psi\cdot(-1)^r$$

$$d\phi\wedge\bar{*}_E\psi + (-1)^{p+q}\phi\,\bar\partial\bar{*}_E\psi$$

$$= (-1)^{r+1}\int\phi\wedge\bar{*}_E(\bar{*}_E\bar\partial\bar{*}_E\psi) \qquad (-1)^r$$

$$= -(\phi,\bar{*}_E\bar\partial\bar{*}_E\psi)_{L^2} = (\phi,\bar\partial^*\psi)_{L^2}$$

$$\Rightarrow \boxed{\bar\partial^*\psi = -\bar{*}_E\bar\partial\bar{*}_E\psi}$$

$\psi = \Sigma\, \eta_\alpha\otimes s_\alpha$

(p,q) holomorphic

Hodge Thm: $A^{p,q}(E) = \underbrace{\mathcal{H}\oplus \mathrm{Im}\,\bar\partial}_{\ker(\bar\partial)}\oplus \underbrace{\mathrm{Im}\,\bar\partial^*}_{\ker(\bar\partial)^\perp}$. $\dim\mathcal{H}<+\infty$.

$$\Rightarrow \mathcal{H}\cong\frac{\ker(\bar\partial)}{\mathrm{Im}(\bar\partial)} = H^{p,q}_{\bar\partial}(E)\cong H^q(X,\Omega^p_X\otimes\mathcal{O}(E))$$

sheaf of hol. p-form

local hol. sec. of E.

Kodaira–Serre duality:

$$H^q(X,\Omega^p(E)) \cong H^{n-q}(X,\Omega^{n-p}(E^*))^*.$$

$$\begin{array}{ccc} \cong & & \cong \\ \mathcal{H}^{p,q}(E) & \xrightarrow{\bar{*}_E} & \mathcal{H}^{n-p,n-q}(E^*). \end{array}$$

$$\bar{*}_E\cdot\Delta_{\bar\partial} = \bar{*}_E\cdot(\underbrace{\bar\partial\bar\partial^*+\bar\partial^*\bar\partial}) = -\bar{*}_E\bar\partial\bar{*}_E\bar\partial\bar{*}_E - (\bar{*}_E)^2\,\bar\partial\bar{*}_E\bar\partial$$

$(\bar{*}_E)^2 = (-1)^{p+q}$

$$(-\bar\partial\bar{*}_E\bar\partial\bar{*}_E - \bar{*}_E\bar\partial\bar{*}_E\bar\partial)$$

$$\Delta_{\bar\partial}\cdot\bar{*}_E = -\bar\partial\bar{*}_E\bar\partial\bar{*}_E^2 - \bar{*}_E\bar\partial\bar{*}_E\bar\partial\bar{*}_E$$

$\bar{*}_E^2 = (-1)^{p+q}$

$(p,q)\xrightarrow{\bar\partial}(p,q+1)$

$\downarrow\bar{*}_E$

$(n-q-1,n-p+1)\xleftarrow{\bar\partial}(n-q-1,n-p)$

$\bar{*}_E^2 = (-1)^{p+q}$

$$H^q(X, \Omega^p(E)) \times H^{n-q}(X, \Omega^{n-p}(E^*)) \to \mathbb{C}$$

$$(\phi \ , \ \psi) \longmapsto \int \phi \wedge \psi$$

$$\shortparallel$$

$$\int \phi \wedge \bar{*}_E(\bar{*}_E \psi)$$

$$\shortparallel$$

$$(\phi, \bar{*}_E \psi)_{L^2}$$

<u>Ex</u>: $H^n(\mathbb{P}^n, \Omega^n) \cong H^0(\mathbb{P}^n, \mathcal{O})^* \cong \mathbb{C}$.

· Kähler mfd. X compact complex mfd.

g: Hermitian metric on TX

$\rightsquigarrow \omega_g = \sqrt{-1} \sum g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}) dz_i \wedge d\bar{z}_j = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$

g is called Kähler if $d\omega_g = 0 \Leftrightarrow \underline{\frac{\partial g_{i\bar{j}}}{\partial z_k}(p) = \frac{\partial g_{k\bar{j}}}{\partial z_i}(p)}$

$\Leftrightarrow \forall\, p \in X$, $\exists$ a holomorphic chart $(\underset{p}{U}, \{z_i\})$ s.t.

$$\underline{\begin{cases} z_i(p) = 0, & i = 1, \cdots, n \\ \text{and} \quad g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2). \end{cases}} \rightsquigarrow \text{Bochner coordinates}$$

$$(a_{i_1 \cdots i_k} z_1^{i_1} \cdots z_n^{i_k} = 0).$$

$$\bigoplus_r \Lambda^r_{\mathbb{C}}(\underbrace{T^*X \otimes \mathbb{C}}) \cong \bigoplus_{p,q} \Lambda^{p,q} T^*X$$

$$\Lambda^r T^*X \otimes \mathbb{C} = \Lambda^r \operatorname{Span}\{dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n\}$$

$$= \bigoplus_{p+q=r} \Lambda^{p,q}$$

$$\Lambda^{p,q} = \operatorname{Span}\{dz_I \wedge d\bar{z}_J\}, \quad dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

$$\underline{sl_2(\mathbb{C}) \circlearrowleft \Lambda^\bullet_{\mathbb{C}} V_{\mathbb{C}}} = \bigoplus_k \underline{\operatorname{Sym}^k(\mathbb{C}^2)}$$

$$sl_2(\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{C}) : \alpha + \delta = 0 \right\}$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$[H, X] = 2X, \quad [X, Y] = H$$

$$[H, Y] = -2Y$$

$$H \mapsto \sum_{r=0}^{2n} (n-r)\pi_r, \qquad X \mapsto L = \omega\wedge, \qquad Y \mapsto L^* = \Lambda$$

$$\omega = \sqrt{-1} \sum_{i=1}^n e_i \wedge \bar{e}_i$$

$$L^*$$

$$(L\eta_1, \eta_2) = (\eta_1, \Lambda \eta_2)$$

$$(L\eta_1, \eta_2) = (\omega\wedge\eta_1, \eta_2)$$

$$[\Lambda, L]\eta_{(p,q)} = \Lambda \cdot \omega \wedge \eta - \omega \wedge \Lambda \eta$$

$$= (n-r)$$

$$L : \Lambda^{p,q} \to \Lambda^{p+1,q+1}$$

$$\Lambda : \Lambda^{p,q} \to \Lambda^{p-1,q-1}$$

$r = p+q$

$\dim_{\mathbb{C}} X = n$.

$$\eta = \eta_0 + \eta_1 + \cdots + \eta_{2n} \xmapsto{H} \sum_{r=0}^{2n} (n-r)\eta_r$$

$\rightsquigarrow$ Lefschetz decomposition: $\varphi \in \Lambda^r V$

$$\varphi = \sum_{k \geq (r-n)^+} L^k \underbrace{\varphi_k}_{\uparrow}$$

primitive $(r-2k)$-form.

$$\Lambda^n V = (\Lambda^n V)_{prim} \oplus L(\underbrace{\Lambda^{n-2} V}_{\substack{\| \\ ((\Lambda^{n-2}V)_{prim} \oplus L\Lambda^{n-4}V)}})$$

$$\Lambda^n V = (\Lambda^n V)_{prim} \oplus \bigoplus_{k \geq 1} L^k (\Lambda^{n-2k} V)_{prim}.$$

<u>Thm</u> (Kähler identities). $d, d^*, \underset{\substack{\| \\ \omega\wedge}}{L}, \Lambda, \overset{\substack{\text{complex} \\ \text{str.} \\ \swarrow}}{J}$

(a) $\underline{[L, d] = 0, \ [L^*, d^*] = 0}$.

(b) $\underline{[L, d^*] = d_c}$, $[L^*, d] = -d_c^*$ where

$$d_c = J^{-1} d J.$$

$$J : \Lambda^{p,q} \longrightarrow \Lambda^{p,q}$$
$$\eta \longmapsto (\sqrt{-1})^{p-q} \cdot \eta$$

$$d = \partial + \bar{\partial}$$

$$\underset{(p,q)}{d_c \eta} = J^{-1} d J \eta = J^{-1} d \cdot \left((\sqrt{-1})^{p-q} \eta \right) = (\sqrt{-1})^{p-q} \cdot J^{-1} \left(\underset{(p+1,q)}{\partial \eta} + \underset{(p,q+1)}{\bar\partial \eta} \right)$$

$$= (\sqrt{-1})^{p-q} \cdot \left(\frac{1}{(\sqrt{-1})^{p+1-q}} \cdot \partial \eta + \frac{1}{(\sqrt{-1})^{p-q-1}} \bar\partial \eta \right)$$

$$= -\sqrt{-1}\, \partial \eta + \sqrt{-1}\, \bar\partial \eta = \sqrt{-1} (\bar\partial - \partial) \eta .$$

$$[L, d^*] = [L, \partial^* + \bar\partial^*] = [L, \partial^*] + [L, \bar\partial^*]$$

$$\rotatebox{90}{=}$$

$$d_c = \sqrt{-1} (\bar\partial - \partial)$$

$$\Leftrightarrow \quad [L, \partial^*] = \sqrt{-1}\, \bar\partial , \quad [L, \bar\partial^*] = -\sqrt{-1}\, \partial .$$

$$[\Lambda, \partial] = \sqrt{-1}\, \bar\partial^* , \quad [\Lambda, \bar\partial] = -\sqrt{-1}\, \partial^*$$

$$\underset{\substack{\rotatebox{90}{=} \\ \partial\Lambda - \Lambda\partial}}{(L\partial^* - \partial^* L)^*} = \underset{\substack{\rotatebox{90}{=} \\ -\sqrt{-1}\, \bar\partial^*}}{(\sqrt{-1}\, \bar\partial)^*}$$

$$0 = \underbrace{\bar\partial \partial^* + \partial^* \bar\partial} = \bar\partial \frac{[\Lambda, \bar\partial]}{-\sqrt{-1}} + \frac{[\Lambda, \bar\partial]}{-\sqrt{-1}} \bar\partial = \sqrt{-1} \cdot \left(\bar\partial \Lambda \bar\partial - \bar\partial \bar\partial \Lambda + \Lambda \bar\partial \bar\partial - \bar\partial \Lambda \bar\partial \right)$$

$$\underline{\Delta_d} = d d^* + d^* d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial)$$

$$= \underbrace{\Delta_\partial + \Delta_{\bar\partial}} + \underset{0}{\underline{(\partial \bar\partial^* + \bar\partial \partial^*)}} + \underset{0}{\underline{(\partial^* \bar\partial + \bar\partial^* \partial)}}$$

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$
$$\| $$
$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial$$

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}.$$

$$\left\{\begin{array}{l} \mathcal{H}_d^r = \bigoplus_{p+q=r} \mathcal{H}_{\bar\partial}^{p,q} = \bigoplus_{p+q=r} \left(\mathcal{H}_d^r \cap A^{p,q}\right) \\ \quad \wr\wr \qquad\qquad\quad \| \\ H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H_{\bar\partial}^{p,q}(X) \\ \eta \in \mathcal{H}_d^{p,q}, \quad \bar\eta \in \mathcal{H}_d^{q,p} \Rightarrow \overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p} \end{array}\right.$$

pure Hodge structure on $H^r(X,\mathbb{Z})$

(For quasi-proj. Kähler mfd. $\exists$ mixed Hodge structure (Deligne)) compact

Ex: X is 1-dim. complex mfd. = Riemann surface

$\omega = \sqrt{-1}\, g\, dz \wedge d\bar{z} \Rightarrow d\omega = 0. \Rightarrow \omega$ is Kähler.

$$\Rightarrow \quad \underbrace{H^1(X, \mathbb{C})}_{\dim_{\mathbb{C}} = 2g} = H^{1,0}(X) \oplus H^{0,1}(X)$$

$H^{1,0}(X) \cong \underline{H^0(\Omega^1)} = \{\text{hol. 1-form}\}$, $\dim_{\mathbb{C}} = g$

$H^{0,1}(X) = H^1(\Omega^0)$

g = genus

$\Rightarrow$ $\underline{\text{genus}} = \underline{\dim H^0(X, \Omega^1)}$.

$\overline{H^{p,q}} = H^{q,p}$

Ex: $\underline{\dim_{\mathbb{C}} H^k(X, \mathbb{C})} = \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X) = \sum_{p+q=k} h^{p,q}$, $\underline{h^{p,q} = h^{q,p}}$

$\dim_{\mathbb{C}} H^k(X, \mathbb{C}) = b_k(X)$

$\Rightarrow$ b_k is even for k odd.

$b_k = 2 \cdot \left(h^{0,k} + h^{1,k} + \cdots + h^{\frac{k-1}{2}, \frac{k+1}{2}}\right)$

$b_1(S^3 \times S^1) = 1 \Rightarrow \underline{S^3 \times S^1}$ has no Kähler structure.

$\exists$ complex structure. Hopf surface.