Leray Thm: $\mathcal{F}$ sheaf. $\mathcal{U} = \{U_i, i \in \mathbb{N}\}$.

$$H^q(U_I, \mathcal{F}) = 0, \quad U_I = \bigcap_{i \in I} U_i, \quad q > 0.$$

Then $H^q(X, \mathcal{F}) = \underline{\check{H}^q(\mathcal{U}, \mathcal{F})}$

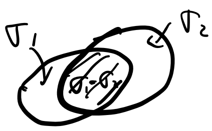
$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^k(\mathcal{U}, \mathcal{F}) \to \cdots$$

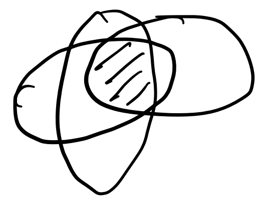
$C^0(\mathcal{U},\mathcal{F}) = \{\sigma_i : i \in I\}$, $\sigma_i \in \Gamma(U_i, \mathcal{F}) = \mathcal{F}(U_i)$

$C^k(\mathcal{U},\mathcal{F}) = \bigoplus_{|I| = k+1} \Gamma(U_I, \mathcal{F})$

$$\delta(\{\sigma_i\}) = \{\sigma_i - \sigma_j : \{i,j\} \in \mathbb{N}^2\}$$

$$\{\sigma_I = \sigma_{i_1 \cdots i_{k+1}} : |I| = k+1\}$$





$$\delta(\{\sigma_{ij}\}) = \sigma_{ij} - \sigma_{ik} + \sigma_{jk} \in \mathcal{F}(U_i \cap U_j \cap U_k)$$

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\delta \uparrow & & \uparrow & & \\
C^1(\mathcal{U}, A^0) & \xrightarrow{d} & C^1(\mathcal{U}, A^1) & \to & \cdots \\
\delta \uparrow & & \uparrow \delta & & \\
C^0(\mathcal{U}, A^0) & \xrightarrow{d} & C^0(\mathcal{U}, A^1) & \to & \cdots
\end{array}$$

$$d\delta = \delta d$$

$$0 \to \mathcal{F} \to A^0 \xrightarrow{d_0} A^1 \to \cdots \to \cdots$$

$$d : C^q(\mathcal{U}, A^p) \to C^q(\mathcal{U}, A^{p+1})$$

$$\bigoplus_{|I| = q+1} \underline{A^p(U_I)} \to \bigoplus_{|I| = q+1} \underline{A^{p+1}(U_I)}$$

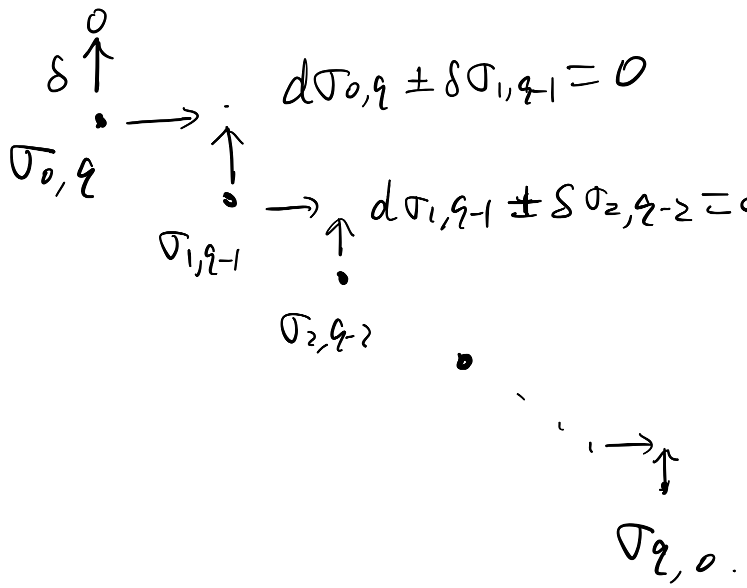
$$\mathcal{E}^l = \bigoplus_{p+q=l} C^q(\mathcal{U}, A^p) \xrightarrow{d + (-1)^p \delta = D} \bigoplus_{p'+q'=l+1} C^{q'}(\mathcal{U}, A^{p'})$$

$$\underline{(d+(-1)^p\delta)}\cdot\underline{(d+(-1)^{p+1}\delta)} = d^2+(-1)^p(d\delta-\delta d)+\delta^2\cdot(-1)^{2p+1}=0$$

$$D^2=0$$

$$\mathcal{E}^0 \xrightarrow{D} \mathcal{E}^1 \xrightarrow{D} \mathcal{E}^2 \to \cdots$$

$$H^q_D(\mathcal{E}^\bullet) = \frac{\operatorname{Ker}(D:\mathcal{E}^q\to\mathcal{E}^{q+1})}{\operatorname{Im}(D:\mathcal{E}^{q-1}\to\mathcal{E}^q)}.$$



$$d\sigma_{0,q}\pm\delta\sigma_{1,q-1}=0$$

$$d\sigma_{1,q-1}\pm\delta\sigma_{2,q-2}=0$$

$$H^q_D(\mathcal{E}^\bullet)\cong H^q_d(\Gamma(A^\bullet))\cong H^q(X,\mathcal{F})$$

$$\begin{array}{c}\wr\\ H^q_\delta(C^\bullet(\mathcal{U},\mathcal{F}))\\ \wr\\ \check{H}(\mathcal{U},\mathcal{F}).\end{array}$$

$$\begin{tikzcd}
C^2(\mathcal{U},\mathcal{F}) \arrow[r] & \cdots & \cdots & \\
C^1(\mathcal{U},\mathcal{F}) \arrow[u, "\delta"] \arrow[r] & C^1(\mathcal{U},A^0) \arrow[u, "\delta"] \arrow[r, "d"] & C^1(\mathcal{U},A^1) \arrow[u] \arrow[r] & \cdots \\
C^0(\mathcal{U},\mathcal{F}) \arrow[u, "\delta"] \arrow[r] & C^0(\mathcal{U},A^0) \arrow[u, "\delta"] \arrow[r, "d"] & C^0(\mathcal{U},A^1) \arrow[u, "\delta"] \arrow[r] & {} \\
 & \Gamma(A^0) \arrow[u] \arrow[r, "\delta"] & \Gamma(A^1) \arrow[u] \arrow[r, "\delta"] & {}
\end{tikzcd}$$

Horizontal:

$$H^k\left(C^q(\mathcal{U}, \underline{A}^0) \to C^q(\mathcal{U}, \underline{A}^1) \to C^q(\mathcal{U}, \underline{A}^2) \to \right) \quad ..$$

$$\bigoplus_{|I|=q+1} H^k(U_I, F) = 0, \quad \text{if } k>0.$$

$$C^q(\mathcal{U}, F) \qquad k=0$$

Vertical:

$$H^k\left(C^0(\mathcal{U}, A^p) \to C^1(\mathcal{U}, A^p) \to \cdots \right)$$

$$\cong \begin{cases} \check{H}^k(\mathcal{U}, A^p) = 0, & k>0 \\ \Gamma(A^p) & k=0 \end{cases}$$

A^p fine $\exists$ ~~partition~~ unity

$$H^q(\mathcal{E}^\cdot, D) = \underline{H_d^q(H_\delta^0)} = H(X,F) \cong \check{H}(\mathcal{U},F)$$

$$\cong \underline{H_\delta^q(H_d^0) = H_\delta^q(C^\cdot(\mathcal{U},F))}$$

Ex: $H^q(\mathbb{P}^1, \mathcal{O})$ $\qquad U = \{z_0 \neq 0,\ u = \frac{z_1}{z_0}\} \cong \mathbb{C}$

$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ $\qquad V = \{z_1 \neq 0,\ v = \frac{z_0}{z_1}\} \cong \mathbb{C}$, $\quad v = u^{-1}$

$[z_0, z_1]$.

$$U \cap V = \mathbb{C}^*$$

$$H^1(\mathbb{C}, \mathcal{O}) = \frac{\mathrm{Ker}(\bar\partial : A^{0,1} \to A^{0,2} = 0)}{\mathrm{Im}(\bar\partial : A^{0,0} \to A^{0,1})} \qquad a\,d\bar z = \left(\frac{\partial}{\partial \bar z}\varphi\right) d\bar z$$

$$\| $$

$$H^1(\mathbb{C}^*, \mathcal{O}) = 0 \iff \frac{\partial \varphi}{\partial \bar z} = a, \ \forall a \text{ complex valued fct.}$$

is solvable

($H^q(\mathbb{C}^k \times (\mathbb{C}^*)^\ell, \mathcal{O}) = 0$, $\forall q \geq 1$. $\bar\partial$ - Poincaré Lemma).

$$H^1(\mathbb{P}^1, \mathcal{O}) \cong \check{H}^1(\mathcal{U}, \mathcal{O}) = \frac{\mathrm{Ker}(\delta : C^1(\mathcal{U}, \mathcal{O}) \to C^2 = 0)}{\mathrm{Im}(\delta : C^0(\mathcal{U}, \mathcal{O}) \to C^1(\mathcal{U}, \mathcal{O}))}$$

$$C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \oplus \mathcal{O}(V) = \left\{ \left(\sum_{n \geq 0} a_n u^n, \sum_{n \geq 0} b_n v^n \right) \right\}$$

$$\downarrow \delta \qquad\qquad \downarrow$$

$$C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V) = \left\{ \sum_{m \in \mathbb{Z}} c_m u^m \right\} \qquad \sum_{n \geq 0} a_n u^n - \sum_{n \geq 0} b_n u^{-n}$$

$U \cap V \cong \mathbb{C}^*$

$$= \sum_{n<0} b_n u^{-n} + (a_0 - b_0) + \sum_{n>0} a_n u^n$$

$$\Rightarrow H^1(\mathbb{P}^1, \mathcal{O}) = 0.$$

$H^1(\mathbb{P}^1, \Omega^1)$

$$C^0(\mathcal{U}, \Omega^1) = \underbrace{\Omega^1(U)}_{du} \oplus \Omega^1(V) = \left\{ \left(\sum_{n\geq 0} a_n u^n\right) du, \left(\sum_{n\geq 0} b_n v^n\right) dv \right\}$$

$$\xrightarrow{\delta}$$

$$\underline{C^1(\mathcal{U}, \Omega^1)} = \underline{\Omega^1(U\cap V)} = \underline{\left\{ \left(\sum_{m\in\mathbb{Z}} c_m \cdot u^m\right) \cdot du \right\}}$$

$$\underline{\left(\sum_{n\geq 0} a_n u^n + \sum_{n\geq 2} b_{n-2} u^{-n} \right) du} = \left(\sum_{n\geq 0} a_n u^n\right) \underline{du} + \left(\sum_{n\geq 0} b_n u^{-n}\right) u^{-2} \underline{du}$$

$v = u^{-1}$

$dv = -u^{-2} du$

$$= \underline{\mathbb{C}\left(u^{-1} \cdot du\right) \cong H^1(\mathbb{P}^1, \Omega^1)}$$

$$\Rightarrow \boxed{H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & \text{if } p = q \leq n \\ 0 & \text{otherwise} \end{cases}}$$

$H^0(\mathbb{P}^n, \Omega^q) = 0$ for $q > 0$. no global holomorphic q-forms

Griffith: $H^p(X, \Omega^q)$ for $X \subset \mathbb{P}^{n+1}$ hypersurface.

<u>Hodge Theory</u>: $\mathcal{E}$ holomorphic vector bundle $\downarrow$ X complex mfd.

$$\bar\partial : A^{0,q}(\mathcal{E}) \to A^{0,q+1}(\mathcal{E})$$

$$\bar\partial\Big(\underbrace{\sum_{\alpha, I} \overset{\text{smooth}}{\phi_{\alpha I}} \overset{\text{hol.}}{S_\alpha} \, \underline{d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_q}}}_{\sigma}\Big) = \sum_{\alpha, I, j} S_\alpha \cdot \frac{\partial \phi_{\alpha I}}{\partial \bar z_j} d\bar z_j \wedge d\bar z_I$$

$I = \{i_1, \dots, i_q\}$, $d\bar z_I = d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_q}$.

$$0 \to \mathcal{O}(\mathcal{E}) \to \underline{A^{0,0}(\mathcal{E}) \to A^{0,1}(\mathcal{E}) \to \cdots}$$

$$\Longrightarrow H^q(\mathcal{O}(\mathcal{E})) \cong H^q_{\bar\partial}(A^{0,\bullet}(\mathcal{E}))$$

Dolbeault. $= H^{0,q}_{\bar\partial}(\mathcal{E})$.

Choose Hermitian metric $\underline{h \text{ on } \mathcal{E}}$

$\underline{g \text{ on } TX}$

$$\|\sigma\|_{L^2}^2 = \int_X |\sigma|^2 \, dv_g$$

$$\sigma = \sum_{\alpha, I} \phi_{\alpha I} \, s_\alpha \, \underline{d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}} \in \Gamma(\underline{A^{0,q}(\mathcal{E})})$$

$$\bar{\partial} : \Gamma(A^{0,q}(\mathcal{E})) \rightarrow \Gamma(A^{0,q+1}(\mathcal{E}))$$

$$\Gamma(A^{0,q}(\mathcal{E})) \xleftarrow{\bar{\partial}^*} \Gamma(A^{0,q+1}(\mathcal{E}))$$

$$(\bar{\partial}\sigma_1, \sigma_2)_{L^2} = (\sigma_1, \bar{\partial}^* \sigma_2)_{L^2}$$

$$\boxed{\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : \Gamma(A^{0,q}(\mathcal{E})) \rightarrow \Gamma(A^{0,q}(\mathcal{E}))}$$

$$\begin{tikzcd} \Gamma(A^{0,q-1}) \arrow[r, "\bar{\partial}"] & \Gamma(A^{0,q}) \arrow[r, "\bar{\partial}"] \arrow[l, bend left, "\bar{\partial}^*"] & \Gamma(A^{0,q+1}) \arrow[l, bend left, "\bar{\partial}^*"] \end{tikzcd}$$

inner product

$$\Delta : V \to V, \quad \Delta^* = \Delta$$

$$\Rightarrow \quad V = \underbrace{\mathrm{Ker}(\Delta)}_{= \mathcal{H}} \oplus \underbrace{\mathrm{Ker}(\Delta)^{\perp}}_{= \mathrm{Im}(\Delta^*) = \mathrm{Im}(\Delta)}$$

$$\Downarrow$$

$$v = v_0 + \Delta \cdot u$$

<u>Thm (Hodge)</u>. $\underline{\Gamma(A^{0,q}(\mathcal{E})) = \underbrace{\mathrm{Ker}(\Delta_{\bar\partial})}_{= \text{harmonic } \mathcal{H}} \oplus_{\perp} \mathrm{Im}(\Delta_{\bar\partial})}$ $\boxed{\dim \mathcal{H} < +\infty}$

$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ $\quad$ $\mathcal{E}$-valued $(0,q)$-form

$$= \mathcal{H} \oplus \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^*, \quad \left(\begin{aligned} &(\bar\partial u, \bar\partial^* v)_{L^2} \\ &\quad \| \\ &(\bar\partial\bar\partial u, v) = 0. \end{aligned} \right)$$

- $\mathrm{Ker}(\bar\partial) = \mathcal{H} \oplus \mathrm{Im}\,\bar\partial$

(Griffith-Harris
Wells)

$$\Rightarrow \quad \underline{H^q_{\bar\partial}(A^{0,\bullet}(\mathcal{E})) = \frac{\mathrm{Ker}(\bar\partial)}{\mathrm{Im}(\bar\partial)} = \mathcal{H}^{0,q}(\mathcal{E}).}$$

$$0 \to \underset{\Omega^p \otimes_{\mathcal{O}_X} \mathcal{O}(\mathcal{E})}{\underset{\parallel}{\Omega^p(\mathcal{E})}} \to A^{p,0}(\mathcal{E}) \to A^{p,1}(\mathcal{E}) \to \cdots$$

$$\Rightarrow \quad H^q(X, \Omega^p(\mathcal{E})) \cong H^q_{\bar\partial}(A^{p,\bullet}(\mathcal{E})) = H^{p,q}_{\bar\partial}(\mathcal{E})$$

$$\underset{\bar\partial\bar\partial^* + \bar\partial^*\bar\partial}{\underset{\parallel}{}}\!\!\mathrm{Ker}(\Delta_{\bar\partial}) = \mathcal{H}^{p,q}(\mathcal{E}) \quad (\cong H^{p,q}_{\bar\partial}(\mathcal{E}))$$

$\uparrow$ finite dim.

$$\sigma = \sum_{\substack{|I|=p \\ |J|=q}} \phi_{\lambda I J}\, s_\lambda \, dz_I \wedge d\bar z_J$$

$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $\quad d\bar z_J = d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}$.

Kodaira-Serre duality: $\left(H^q(X, \Omega^p(\mathcal{E}))\right)^* = H^{n-q}(X, \Omega^{n-p}(\mathcal{E}^*))$

$$H^{\overset{q}{\shortparallel}}(\mathbb{P}^{\overset{n}{\shortparallel}}, \Omega^{\overset{1}{\shortparallel}}) \longleftrightarrow H^0(\mathbb{P}^1, \mathcal{O}) \cong \mathbb{C}.$$