$\mathcal{F}$ sheaf of abelian groups

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \xrightarrow{d_1} \mathcal{E}^2 \to \cdots$$

resolution: $\mathrm{Ker}(d_0) = i(\mathcal{F})$, $\quad \mathrm{Ker}(d_{i+1}) = \mathrm{Im}(d_i)$, $i \geq 0$.

$$\Longleftrightarrow \quad \mathcal{F} \underset{\text{quasi-iso}}{\cong} \left(0 \to \mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \to \mathcal{E}^2 \to \cdots\right)$$

$$\begin{array}{ccccccccc} & & & & 0 & & 1 & & \\ & & & & \downarrow d_0 & & \downarrow d_1 & & \\ 0 & \to & 0 & \to & \mathcal{F} & \to & 0 & \to & 0 \end{array}$$

cohomology sheaves $\quad \mathcal{H}^k(\mathcal{F}) = \begin{cases} \mathcal{F} & k=0 \\ 0 & k \neq 0 \end{cases}$

$$\mathcal{H}^k(\mathcal{F}) = \frac{\mathrm{Ker}(d_k)}{\mathrm{Im}(d_{k-1})}$$

$$\mathcal{H}^k(\mathcal{E}^\bullet) \cong \begin{cases} \mathcal{F} & k=0 \\ 0 & k \neq 0 \end{cases}$$

Def: Choose an injective solution

$$\mathcal{F} \to \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \mathcal{I}^2 \to \cdots$$

$\mathcal{I}^k$ injective: For every injective morphism

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "\phi"'] & B \arrow[d, "\exists \psi"] \\ & \mathcal{I} \end{tikzcd}$$

$\mathcal{I}^k = \mathcal{I}$

Define:
$$\boxed{R^k\Gamma(\mathcal{F}) = H^k(\mathcal{F}) = \frac{\mathrm{Ker}(d_k : \Gamma(\mathcal{I}^k) \to \Gamma(\mathcal{I}^{k+1}))}{\mathrm{Im}(d_{k-1} : \Gamma(\mathcal{I}^{k-1}) \to \Gamma(\mathcal{I}^k))}.}$$

$$\Gamma(\mathcal{I}) = \mathcal{I}(X), \quad H^0(\mathcal{I}) = \mathcal{I}(X).$$

$\Gamma: \mathcal{F} \longrightarrow \Gamma(\mathcal{F})$ left-exact.

$\{\text{Sheaves}\} \quad \{\text{Abelian groups}\}$

Fact: $0 \to A \to B \to C \to 0$ short exact sequence

induces long exact sequence of coh. groups:

$$\cdots \to H^k(A) \to H^k(B) \to H^k(C) \to H^{k+1}(A) \to H^{k+1}(B) \to \cdots$$

$\Downarrow$ Prove by induction

Thm (Abstract De Rham Thm).

$$0 \to \mathcal{F} \to \underbrace{\mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \to \mathcal{E}^2 \to \cdots}_{\mathcal{E}^i \text{ acyclic}}$$

Assume $H^q(\mathcal{E}^i) = 0,\ q > 0.$

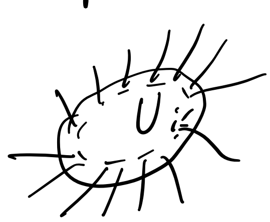
$$H^k(\mathcal{F}) = \frac{\ker(d_k : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1}))}{\mathrm{Im}(d_{k-1} : \Gamma(\mathcal{E}^{k-1}) \to \Gamma(\mathcal{E}^k))}.$$

Def: $\mathcal{F}$ is called a fine sheaf if

For any locally finite open cover $\{U_i\}$ of X, $\exists$ a family of sheaf morphisms $\eta_i : \mathcal{F} \to \mathcal{F}$ s.t.

(a) $\sum \eta_i = 1$ (b) $\eta_i(\mathcal{F}_x) = 0$ $\forall x$ in some nbhd. of the complement of U_i

($\exists$ partition of unity)



Ex: $\mathcal{C}_X$ sheaf of continuous fcts.

$\mathcal{C}^k$ ... C^k fcts.

$\mathcal{A}$ ... sm. fcts.

R fine sheaf of rings, any module of R-sheaves is a fine sheaf

$\mathcal{A}^k$ ... sm. k-form

$\mathcal{A}^{p,q}$ ... sm. (p,q)-forms on complex mfds.

Lem: Fine sheaves are acyclic.

$$F \text{ is fine} \Rightarrow H^q(X,F)=0, \quad q>0.$$

Ex: $0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$

$$\Rightarrow H^k(X,\mathbb{R}) = \frac{\operatorname{Ker}(d_k : A^k(X) \to A^{k+1}(X))}{\operatorname{Im}(d_{k-1} : A^{k-1}(X) \to A^k(X))}.$$

(De Rham Thm)

Ex: $0 \to \mathcal{E} \to A^{0,0}(\mathcal{E}) \xrightarrow{\bar\partial} A^{0,1}(\mathcal{E}) \xrightarrow{\bar\partial} \cdots \to A^{0,k}(\mathcal{E}) \to \cdots$

sheaf of holomorphic sections

Sm (0,k)-form $\mathcal{E}$-valued.

$$\sum_{i_1,\dots,i_k,\alpha} f^{\alpha}_{i_1\cdots i_k} s_\alpha \, d\bar z_{i_1}\wedge\cdots\wedge d\bar z_{i_k}$$

Local hol. nonvanishing section of s.

Dolbeault Thm.

f^α is smooth $\mathbb{C}$ valued.

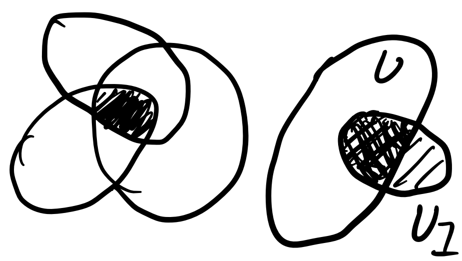
$$\Rightarrow \quad H^q(\mathcal{E}) \cong \frac{\ker\left(\bar\partial : \Gamma(A^{0,q}(\mathcal{E})) \to \Gamma(A^{0,q+1}(\mathcal{E}))\right)}{\mathrm{Im}\left(\bar\partial : \Gamma(A^{0,q-1}(\mathcal{E})) \to \Gamma(A^{0,q}(\mathcal{E}))\right)}.$$

Čech resolution: $\mathcal{F}$ a sheaf on X.

$\mathfrak{U} = \boxed{\{U_i : i \in \mathbb{N}\} \text{ countable covering by open subsets of } X}$

Set $U_I = \bigcap_{i\in I} U_i$, $I \subset \mathbb{N}$.

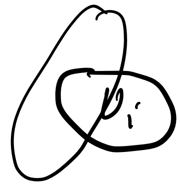
$\mathcal{F}_I = (j_I)_*(\mathcal{F}|U_I)$



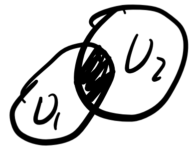
$\mathcal{F}_I(U) = (\mathcal{F}|U_I)(U \cap U_I)$.

$$F^k = \bigoplus_{|I|=k+1} F_I$$

$$F^0 = \bigoplus_{|I|=1} F_I$$



$$F^1 = \bigoplus_{|I|=2} F_I$$

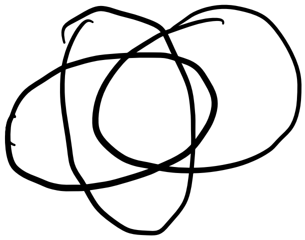


$d: F^k \to F^{k+1}$

$$(d\sigma)_{j_0 \cdots j_{k+1}} = \sum_i (-1)^i \sigma_{j_0 \cdots \hat{j_i} \cdots j_{k+1}}|_{U \cap U_{j_0 \cdots j_{k+1}}}$$

$$0 \to F^0 \xrightarrow{d} F^1 \xrightarrow{d} \cdots F^n \xrightarrow{d} F^{n+1}$$

$$\frac{\ker(d: \Gamma(F^k) \to \Gamma(F^{k+1}))}{\mathrm{Im}(d: \Gamma(F^{k-1}) \to \Gamma(F^k))} = \check{H}(\mathcal{U}, F)$$



$U_{j_0} \cap \cdots \cap U_{j_{k+1}} = U_I$

$\{(\sigma_{j_0 j_1 \cdots j_{k+1}}) : j_0 j_1 \cdots j_{k+1}\}$

$k=0$: $\{\sigma_i \in F(U_i)\} = \sigma$

$$d\sigma = \{(\sigma_i - \sigma_j)|_{U_i \cap U_j} \in F(U_i \cap U_j)\}$$

If $d\sigma = 0$, $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$, then $\exists\, \sigma \in \Gamma(X)$ s.t. $\sigma|_{U_i} = \sigma_i$

$k=1$: $\{\sigma_{ij} \in F(U_i \cap U_j)\} = \sigma$

$$(d\sigma)_{ijk} = \{\sigma_{ij} - \sigma_{ik} + \sigma_{jk} \in F(U_i \cap U_j \cap U_k)\}$$

<u>Thm</u> (Leray): If $H^q(U_I, F) = 0$, $\forall q > 0$, $I \subset N$.

then $\underline{H^q(X, F) = \check{H}^q(\mathcal{U}, F)}$.

<u>Ex</u>: $F = \mathcal{O}$ sheaf of holomorphic ~~fcts~~.

For $q>0$, $\underline{H^q(U, \mathcal{O}) = 0}$ when <u>U is biholomorphic to a convex bounded domain in $\mathbb{C}^n$</u>.

$$\shortparallel$$

$$\frac{\operatorname{Ker}(\bar\partial : A^q(U) \to A^{q+1}(U))}{\operatorname{Im}(\bar\partial : A^{q-1}(U) \to A^q(U))}$$

Assume $H^q(U_I, \mathcal{O}) = 0$.

$$0 \to \mathcal{O} \to A^{0,0} \xrightarrow{\bar\partial} A^{0,1} \xrightarrow{\bar\partial} A^{0,2} \to \cdots$$

$$0 \to Z^{0,k} \xrightarrow{\bar\partial} A^{0,k} \xrightarrow{\bar\partial} Z^{0,k+1} \to 0$$

$$Z^{0,k+1} = \bar\partial A^{0,k}$$

$$
\begin{tikzcd}
\check{H}^q(\mathcal{U}, Z^{0,k}) \arrow[r] & \check{H}^q(\mathcal{U}, \overset{\text{fine}}{A^{0,k}}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \check{H}^q(\mathcal{U}, Z^{0,k+1}) \arrow[d, "\wr"] \\
& 0 & \\
& \check{H}^{q+1}(\mathcal{U}, A^{0,k}) \arrow[d, phantom, "\shortparallel"] & \check{H}^{q+1}(\mathcal{U}, Z^{0,k}) \arrow[l] \\
& 0 &
\end{tikzcd}
$$

Fact: For any fine sheaf A, $\check{H}^q(\mathcal{U}, A) = 0$, $q > 0$.

$$\Rightarrow \quad \check{H}^{q+1}(\mathcal{U}, Z^{0,k}) \cong \check{H}^q(\mathcal{U}, Z^{0,k+1})$$

$$\cong \cdots \cong \check{H}^1(\mathcal{U}, Z^{0,q+k})$$

$k=0$: $\quad \check{H}^q(\mathcal{U}, \mathcal{O}) \cong \check{H}^1(\mathcal{U}, Z^{0,q-1})$.

$q+1 \to q$

$$0 \to Z^{0,q-1} \xrightarrow{\bar\partial} A^{0,q-1} \xrightarrow{\bar\partial} Z^{0,q} \to 0$$

$$
\begin{tikzcd}
\check{H}^0(A^{0,q-1}) \arrow[r, "\bar\partial"] & \check{H}^0(Z^{0,q}) \arrow[r] & \check{H}^1(Z^{0,q-1}) \arrow[r] & \check{H}^1(A^{0,q-1}) \arrow[d, phantom, "\shortparallel"] \\
& & & 0
\end{tikzcd}
$$

$$\check{H}^1(Z^{0,q-1}) = \frac{\check{H}^0(Z^{0,q})}{\bar\partial \check{H}^0(A^{0,q-1})} = \frac{H^0(Z^{0,q})}{\bar\partial H^0(A^{0,q-1})} = H^q_{\bar\partial}(\mathcal{O}) \cong H^q(\mathcal{O}).$$

General case:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
\mathcal{F}^2 & & & \\
\mathcal{F}^1 \arrow[u] \arrow[r] & A^{0,1} \arrow[u, "\delta_1"] \arrow[r] & A^{1,1} \arrow[u, "\delta_1"] \arrow[r] & A^{2,1} \arrow[u] \\
\mathcal{F}^0 \arrow[u] \arrow[r] & A^{0,0} \arrow[u, "\delta_0"] \arrow[r, "d_0"] & A^{1,0} \arrow[u, "\delta_1"] \arrow[r, "d_1"] & A^{2,0} \arrow[u] \\
\mathcal{F} \arrow[u] \arrow[r] & A^0 \arrow[u] \arrow[r, "d_0"] & A^1 \arrow[u] \arrow[r, "d_1"] & A^2 \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

$\mathcal{F} = \mathbb{R}$

Čech resolution

fine sheaf.

$$A^{2,1} = \bigoplus_{|J|=2} (i_J)_* (A^2|_{U_J}).$$

Double complex is a fine resolution of $\mathcal{F}$.

$$H^k(\mathcal{F}) = \boxed{\begin{array}{c} \underline{H^k(A^{\bullet,\bullet}, D)} \quad D = d + (-1)^{\bullet}\cdot\delta \\ \parallel \\ \check{H}^k(\mathcal{F}^\bullet) \end{array}}$$

(Bott-Tu)

$$
\begin{tikzcd}
A^{p,q+1} \arrow[r, "d"] & A^{p+1,q+1} \\
A^{p,q} \arrow[u, "\delta"] \arrow[r, "d"] & A^{p+1,q} \arrow[u, "\delta"']
\end{tikzcd}
$$

$$\delta \circ d = d \circ \delta$$

$$\Longleftrightarrow \quad \big(\underbrace{d + (-1)^{q}\delta}_{D}\big) \circ \big(\underbrace{d + (-1)^{q+1}\delta}_{D}\big) = 0.$$