Sheaves $\quad X$ topological space

$\mathcal{F}$ presheaf of Abelian groups: $\underset{\text{open subset}}{U} \longmapsto \mathcal{F}(U)$ abelian group

- $V \subset U \rightsquigarrow \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ homomorphism of Abelian gps.
  $\sigma \mapsto \sigma|_V$ ($\sigma \in \mathcal{F}(U)$, $\sigma|_V \in \mathcal{F}(V)$)

- $W \subset V \subset U \rightsquigarrow \rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(W)$

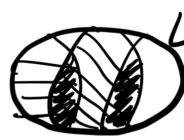
$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr] \arrow[dr] & & \mathcal{F}(W) \\ & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$

Def: sheaf of Abelian gps. is a presheaf satisfying:

for every open subset U of X and every open covering of $U = \bigcup_{V \in \mathcal{V}} V$

$$\mathcal{F}(U) \longrightarrow \prod_{V \in \mathcal{V}} \mathcal{F}(V)$$

$U \subset X$



$$\{(\sigma_V)_{V \in \mathcal{V}} \mid \sigma_V|_{W \cap V} = \sigma_W|_{W \cap V}, \forall V, W \in \mathcal{V}\}$$

morphism of presheaf $\phi : \mathcal{F} \to \mathcal{G}$ s.t.

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

$$\forall \sigma \in \mathcal{F}(U),\ V \subset U \quad \phi_U(\sigma)|_V = \phi_V(\sigma|_V).$$

Lem: For every presheaf $\mathcal{F}$ X, $\exists$ a unique sheaf $\mathcal{F}_f$ over X s.t. (sheafication)

- $\exists$ morphism of presheaf $\phi : \mathcal{F} \to \mathcal{F}_f$.
- $$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall"] \arrow[d, "\phi"'] & \mathcal{G} \text{ « sheaf} \\ \mathcal{F}_f \arrow[ur] & \end{tikzcd}$$

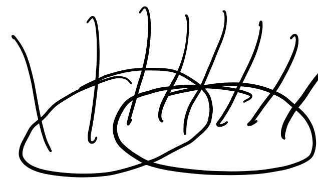
Ex: sm. mfd. X

sheaf of rings $\rightsquigarrow$ $\underline{C}^k$: sheaf of differentiable C^k fcts.

$\rightsquigarrow$ $\underline{A}$: - - - - - - - Smooth fcts.

smooth vector bundle $E = \coprod_{\alpha} U_\alpha \times \mathbb{C}^r / (x, v) \sim (x, g_{\alpha\beta}(x)(v))$.

sheaf of modules over A $\rightsquigarrow$ $\mathcal{E}(U) = \left\{ \begin{array}{l} \text{smooth sections of } E \\ \text{over } U \end{array} \right\}$



· X complex mfd. $E \to X$ hol. vector bdle.

sheaf of ring $\to$ $\mathcal{O}_X(U) = \{$ holomorphic fcts. on $U \}$

sheaf of $\mathcal{O}_X$-module $\rightsquigarrow$ $\mathcal{O}_X(E)(U) = \{$ holomorphic sections on $U \}$.

$$\underset{\mathcal{O}_X}{\underset{\Uparrow}{f}} \cdot \underset{\mathcal{O}_X(E)}{\underset{\Uparrow}{s}} \in \mathcal{O}_X(E).$$

$\Omega^p(U) = \{$ holomorphic p-forms on $U \}$

$A^p(U) = \{$ Smooth p-forms on $U \}$

$A^{p,q}(U) = \{$ smooth (p,q)-forms on $U \}$

$= \{ \alpha = \sum_I \alpha_{I,J} \, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \}$ locally at $p \in U$

· G abelian group.

$U \mapsto G(U) = G$, $\rho_{UV} = Id : G \to G$

$\{g \in G\}$

$g_1 \in G(X_1)$ $g_2 \in G(X_2) = G$

$X = X_1 \cup X_2$

$g_1 \neq g_2$. No $g \in G(X) = G$

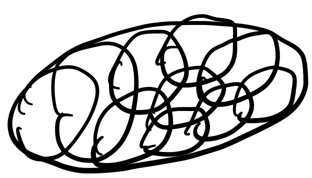
$g|_{X_1} = g_1$, $g|_{X_2} = g_2$

Presheaf F. For each open covering $\mathcal{V}$ of U

$$A_{\mathcal{V}}(U) = \{ (\sigma_V)_{V \in \mathcal{V}}, \sigma_V \in F(V) : \sigma_V|_{W \cap V} = \sigma_W|_{W \cap V}, \forall V, W \in \mathcal{V} \}$$

$\mathcal{V}'$ finer than $\mathcal{V}$

if $\forall V' \in \mathcal{V}', \exists V \in \mathcal{V}$ s.t. $V \subset V'$



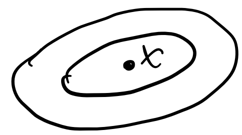
$\rho_{\mathcal{V}, \mathcal{V}', \sigma} : A_{\mathcal{V}'} \longrightarrow A_{\mathcal{V}}$.

$F^{+}(U) = \varinjlim A_{\mathcal{V}}$

• **Def:** Stalk $\mathcal{F}_x$ of a presheaf $\mathcal{F}$ over X at $x \in X$: $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) = \{$ germs of sections of $\mathcal{F}$ at $x \}$

$$\sigma_U \in \mathcal{F}(U), \quad \sigma_V \in \mathcal{F}(V)$$

$$\sigma_U \sim \sigma_V$$



$$\Uparrow$$

$$\sigma_U|_W = \sigma_V|_W, \quad x \in W \subset U \cap V$$

Def: $\phi: \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if $\phi_x: \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Ex: $0 \in \mathcal{F}$ = sheaf of continuous complex-valued fct.

$1 \in \mathcal{G}$ = multiplicative sheaf of continuous no-where zero fcts.

$$\mathcal{F} \xrightarrow{\overset{\phi_U}{\exp(2\pi i \cdot)}} \mathcal{G}$$

$$f \mapsto \exp(2\pi i f).$$

$$\ker(\phi) = \mathbb{Z}$$

$X = \mathbb{C}^*$. $\quad \phi : \mathcal{F}(\mathbb{C}^*) \to \mathcal{G}(\mathbb{C}^*)$ not surjective

$$\begin{array}{cc} \log z & \cup\!\!\!\! \vert \\ & z \end{array}$$

$\phi : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.

Lem: $\phi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves.

$\ker \phi : U \longmapsto \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ is a sheaf.

$\ker \phi = 0$ iff ϕ is injective (i.e. $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ inj. $\forall x \in X$.)

Lem: $\operatorname{Im} \phi =$ sheaf associated to the presheaf

$$U \mapsto \operatorname{Im}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

$\operatorname{Im} \phi = \mathcal{G}$ iff ϕ is surjective.

$\operatorname{Coker}(\phi) =$ sheaf associated to the presheaf

$$U \mapsto \operatorname{Coker}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

- $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ exact (in the middle) if

$$\ker(\psi) = \underline{\mathrm{Im}(\phi)}$$

- Complex of sheaves

$$\to F^{i} \xrightarrow{d_i} F^{i+1} \xrightarrow{d_{i+1}} F^{i+2} \to \cdots$$

$$\underline{d_{i+1} \circ d_i = 0 . \quad \forall\, i \in \mathbb{Z} .}$$

- resolution of a sheaf F : exact complex:

$$0 \to F \xrightarrow{j} F^{0} \xrightarrow{\phi_0} F^{1} \xrightarrow{\phi_1} F^{2} \to \cdots = F^{\bullet}$$

injective

$$\ker(\phi_0) = j(F), \qquad \ker(\phi_{i+1}) = \mathrm{Im}(\phi_i)$$

(De Rham)

Ex: $\mathbb{R}$ Poincaré Lemma

$$0 \to \mathbb{R} \to A^{0} \xrightarrow{d} A^{1} \xrightarrow{d} \cdots \to A^{k} \to \cdots = A^{\bullet}$$

$A^{0} = \{\text{sm. fcts}\}$, $A^{k} = \{\text{sm. } k\text{-forms}\}$

(Dolbeault).

Ex: resolution of $\mathcal{O}_X(E)$

$$0 \to \mathcal{O}_X(E) \hookrightarrow A^{0,0}(E) \xrightarrow{\bar\partial} A^{0,1}(E) \xrightarrow{\bar\partial} \cdots A^{0,k}(E) \to A^{0,n}(E)$$

$\mathcal{O}_X(E) = \{\text{holomorphic sections}\}$, $A^{0,0}(E) = \{\text{sm. sections}\}$, $A^{0,k}(E) = \{\text{sm. } E\text{-valued } (0,k)\text{-forms}\}$

$$\bar\partial\Big(\sum_{i_1,\dots,i_k} s f \, d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_k}\Big) = s \sum_{i_1,\dots,i_k} \bar\partial f \wedge d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_k}$$

· Cohomology of Sheaf $\mathcal{F}$

injective resolution

$$\mathcal{F} \longrightarrow \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \cdots$$

$$H^q(X, \mathcal{F}) = \frac{\ker(d_q : \mathcal{F}^q(X) \to \mathcal{F}^{q+1}(X))}{\operatorname{Im}(d_{q-1} : \mathcal{F}^{q-1}(X) \to \mathcal{F}^q(X))}.$$

is independent of injective resolution.

<u>Thm</u> (Abstract De Rham Thm).

$$0 \to F \to \underbrace{G^0 \xrightarrow{d_0} G^1 \xrightarrow{d_1} \cdots}_{= G^\cdot}$$

Assume that G^i are <u>acyclic</u> in the sense that $\boxed{H^q(X, G^i) = 0 \quad \forall q \geq 1.}$

<u>Then</u>: $$H^q(X, F) = \frac{\ker\left(d_i : \underset{\overset{\shortparallel}{G^i(X)}}{\Gamma(G^i)} \to \Gamma(G^{i+1})\right)}{\operatorname{Im}\left(d_{i-1} : \Gamma(G^{i-1}) \to \Gamma(G^i)\right)}$$

$$0 \to \underset{\overset{\cup\!\shortmid}{\ker d_{i-1}}}{K_i} \longrightarrow G^i \longrightarrow \underset{\overset{\shortparallel}{\operatorname{Im} d_i = \ker d_{i+1}}}{K_{i+1}} \to 0$$

$$0 \to F \to G^0 \to K_1 \to 0$$

$$H^{q-1}(K_1) \xrightarrow{\cong} H^q(F) \to H^q(G^0) \to H^q(K_1)$$

$$H^q(F) \cong H^{q-1}(K_1)$$

$$0 \to K_1 \to G^1 \to K_2 \to 0$$

$$H^q(F) \cong H^{q-1}(K_1) \cong H^{q-2}(K_2)$$

$$\cong \cdots \cong H^1(K_{q-1}).$$

$$K_{q-1} \to G^{q-1} \to K_q$$

$$\boxed{\frac{\ker\left(H^0(G^q) \xrightarrow{d_q}\right)}{\operatorname{Im}\left(H^0(G^{q-1}) \xrightarrow{d_{q-1}}\right)}} = H^1(K_{q-1})$$

$$H^0(K_{q-1}) \to \underline{H^0(G^{q-1})} \to \underline{H^0(K_q)} \to \underline{H^1(K_{q-1})} \to H^1(G^{q-1}) = 0$$

$$H^0(K_q) = \ker\left(H^0(G^q) \to H^0(G^{q+1})\right).$$

$$\underline{0 \to 0 \to F \to 0 \to 0 \to \cdots}$$

$$\| $$

$$F \rightsquigarrow H^q(X, F)$$

$$\underline{0 \to 0 \to F^0 \to F^1 \to \cdots = F^\bullet}$$

$$\rightsquigarrow \mathbb{H}^q(F^\bullet)$$

- If $F^\bullet$ is a resolution of F, then

$$\boxed{\mathbb{H}^q(F^\bullet) = H^q(X, F)}$$

$$\|$$

If F^q is acyclic, i.e. $H^k(F^q) = 0$ $\forall k \geq 1$.

$$\frac{\ker(d_q : H^0(F^q) \to H^0(F^{q+1}))}{\operatorname{Im}(d_{q-1} : H^0(F^{q-1}) \to H^0(F^q))}$$

- $F^\bullet \overset{q\text{-iso}}{\cong} G^\bullet$ if $\mathcal{H}^q(F^\bullet) = \mathcal{H}^q(G^\bullet)$ $\forall q \geq 0$.