Normal Varieties

X irred. affine. X is normal if $k[X]$ is integrally closed.

quasi-proj. X is normal if $\forall x \in X$ has normal affine nbhd.

Fact: · A nonsingular variety is normal.

· $\mathrm{Codim}\, X^{sing} \geq 2$ (regular in codim. 1)

$\Rightarrow$ For algebraic curves, normal $\Leftrightarrow$ nonsingular

X irred. var.. A normalization is an irreducible normal variety X^ν together with a regular map $\nu : X^\nu \to X$ s.t. ν is finite and birational.

If $X = \bigcup_i X_i$ reducible, then define $X^\nu = \bigsqcup_i X_i^\nu$.

Thm: An affine irred. variety X has a normalization that is also affine.

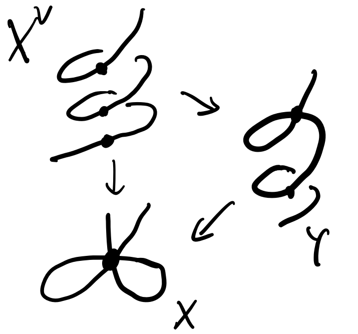
Pf: $k[X] \subset \overline{k[X]} \subset k(X)$, $\overline{R} = \overline{k[X]}$

integral closure $= \{ f \in k(X) : f \text{ is integral over } k[X] \}$

$\overline{R}$ is f.g. k-algebra, and is integral over $k[X]$

$\rightsquigarrow$ $X \leftarrow X^\nu \leftarrow$ affine variety corresponding to $\overline{R} = k[T]/I$. ∎

Thm: (i) If $g: Y \to X$ finite birational map,



$$\begin{tikzcd} & X^\nu \arrow[dl, "h"'] \arrow[dr, "\nu"] & \\ Y \arrow[rr, "g"'] & & X \end{tikzcd}$$

$\uparrow$ finite birational

$\nu = g \circ h$.

Pf: $g: Y \to X \iff k[X] \underset{\text{integral}}{\hookrightarrow} k[Y] \hookrightarrow k(Y) = k(X)$.

$\Rightarrow \quad k[Y] \subset \overline{k[X]}$

$\Rightarrow \quad Y \xleftarrow{h} X^\nu$

(ii) $g: Y \to X$ regular, $g(Y)$ is dense in X and Y is normal

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "g"] & \\ X^\nu \arrow[rr] & & X \end{tikzcd}$$

$\swarrow$ normal

Pf: $\overset{g(Y)\text{ dense}}{k[X] \underset{g^*}{\hookrightarrow} k[Y]}$

$\hookrightarrow \overline{k[X]} \ni u$

$\overline{k[X]} = k[X^\nu]$

u integral over $k[X]$

$u \in k(X) \hookrightarrow k(Y)$.

$\Rightarrow$ u integral over $\underset{\underset{k[X]}{\cup}}{k[Y]}$

$\underbrace{\overline{k[Y]} = k[Y]}_{\Uparrow} \Rightarrow u \in k[Y] \Rightarrow \overline{k[X]} \subset k[Y]$

Y is normal.

$\Rightarrow \quad X^\nu \leftarrow Y$

<u>Cor</u>: The normalization of a variety is unique.

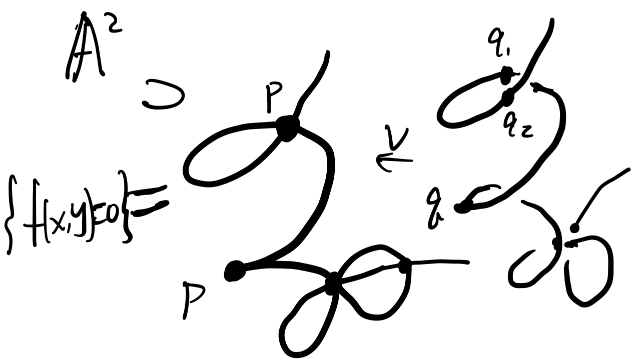
$$\begin{tikzcd} X^{\nu} \arrow[rr, shift left, "\cong"] \arrow[dr, "\nu"'] & & X^{\nu'} \arrow[ll, shift left] \arrow[dl, "\nu'"] \\ & X & \end{tikzcd}$$

<u>For curves</u>:

Thm: An irreducible quasi-proj. curve X has a normalization that is also quasi-projective.

<u>Thm</u>: <u>The normalization of a projective curve is projective</u>

<u>Cor</u>: An irreducible alg. curve is birational to a nonsing. projective curve.



$\nu^{-1}(p) = \{q_1, \dots, q_k\}$

q_i : branches of the curve X at p.

$q \in \underline{X^{\nu}}$ one branch, <u>X^{ν} normal = nonsingular</u>, t local parameter at q.

$$\begin{cases} x = \overset{\neq 0}{a_n} \cdot t^n + a_{n+1} t^{n+1} + \cdots = \tau^n \\ y = b_m \cdot t^m + a_{m+1} t^{m+1} + \cdots \end{cases}$$

power series expansion of $\nu^* x$, $\nu^* y$

$$\Rightarrow \quad \underline{\tau = r_1 t + r_2 t^2 + \cdots}$$

$$= (a_n t^n + a_{n+1} t^{n+1} + \cdots)^{\frac{1}{n}} = t\,(a_n + a_{n+1} t + \cdots)^{\frac{1}{n}}$$

$$\Rightarrow \quad \begin{cases} x = \tau^n \\ y = c_m \tau^m + c_{m+1} \tau^{m+1} + \cdots = c_m x^{m/n} + c_{m+1} x^{\frac{m+1}{n}} + \cdots \end{cases}$$

<u>Newton–Puiseux expansion of y</u>

- $0 \in \mathbb{C}^2$, $\quad C = \{ f(x,y) = 0 \}$

$\mathbb{C}\{x,y\}$ is a UFD. $\quad f(x,y) = f_1 \cdots f_k$.

f_i irreducible branch.

$$f \rightsquigarrow f(x,y) \cdot u(x,y) = y^k + a_1(x) \cdot y^{k-1} + \cdots + a_k(x)$$

$$u(0,0) \neq 0 \qquad a_i(x) \in \mathbb{C}\{x\}.$$

Assume f is irreducible.

$\Rightarrow$ • $\exists\, \rho > 0$, s.t. $\forall\, x \in \mathbb{C}$ with $|x| < \rho$, $x \neq 0$,

The roots $y = y(x)$ to $f(x, y(x)) = 0$ are simple.

k-roots.

- covering map $X - P \to D_\rho - 0$

and deck transformation is transitive.

$\Rightarrow \quad \nu : D_\rho \longrightarrow X$ one-to-one

$t \mapsto (t^k, y_i(t^k))$

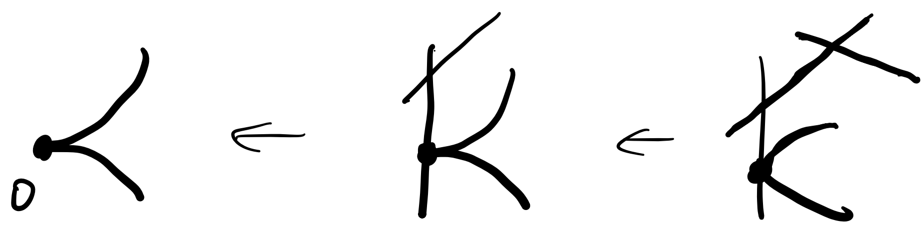
Ex: cusp: $x^p - y^q = 0$ $\quad p < q$

$t \mapsto x = t^q,\ y = t^p = (t^q)^{\frac{1}{p}}$

$\Uparrow$

D_ε

Fact: Every plane algebraic curve can be resolved by blowing up points.



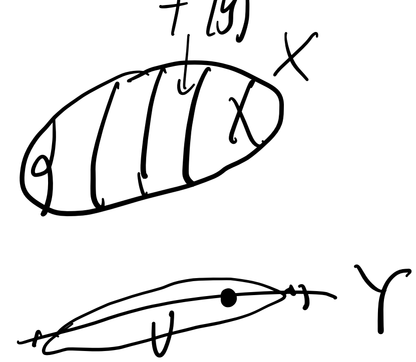
resolution graph

$\Updownarrow$

Puiseux series.

Thm (Bertini Thm) $f: X \to Y$ is regular map. Char $= 0$.

$f(X)$ dense in Y. Assume X is nonsingular. Then there exists a dense open subset $U \subset Y$ s.t. $f^{-1}(y)$ is nonsingular $\forall\, y \in U$.



Assume $f(X) = Y$ and Y is nonsingular.

Lemma: $f^{-1}(y)$ nonsingular if

$d_x f: \Theta_{X,x} \longrightarrow \Theta_{Y,y}$ is surjective for all points $x \in f^{-1}(y)$

Pf:
$$\Theta_{f^{-1}(y),x} \subseteq \ker(d_x f : \Theta_{X,x} \to \Theta_{Y,y}).$$

$m_{Y,y}/m_{Y,y}^2$

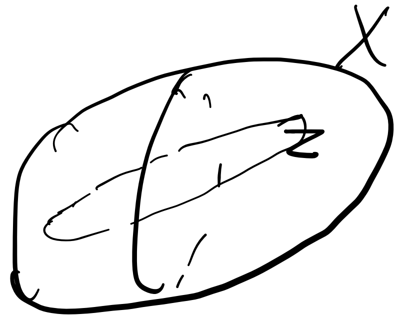
$$\Rightarrow \underbrace{\dim \Theta_{f^{-1}(y),x}}_{\geq \dim f^{-1}(y)} \leq \dim \ker(d_x f) = \dim \Theta_{X,x} - \dim \Theta_{Y,y} = \underbrace{n - m}_{\leq \dim f^{-1}(y)}.$$

$$\Rightarrow \dim \Theta_{f^{-1}(y),x} = \dim f^{-1}(y)$$

$\Rightarrow$ x is nonsingular point of $f^{-1}(y)$.

Lemma: $\exists$ nonempty open subset $V \subset X$ s.t.

$d_x f$ is surjective for $x \in V$



Pf of the Thm:

$Z = \{x \in X : d_x f \text{ is not surjective}\}$

$\uparrow$ set of critical points $\qquad \downarrow$ closed

Need to prove $f(Z)$ is contained in a proper closed subset of Y.

$\uparrow$ set of critical values.

If not, then $f(Z)$ is dense in Y.

$f : Z \to f(Z)$ $\Rightarrow$ $\exists$ open subset $V \subset Z$ s.t.

$f(Z) \underset{\text{dense}}{\subset} Y$

$d_x f : \Theta_{Z,x} \twoheadrightarrow \Theta_{Y,y}$ surjective

$\Theta_{Z,x} \subset \Theta_{X,x}$, $\Theta_{X,x} \twoheadrightarrow \Theta_{Y,y}$

contradict that $x \in Z$

i.e. $d_x f$ not surj.

Now choose $U = Y \setminus \overline{f(Z)}$.

$\Rightarrow f^{-1}(y)$ is nonsingular

because $f^{-1}(y) \cap Z = \phi$ i.e. $d_x f$ is surjective for any $x \in f^{-1}(y)$.

open subset $U \subset X^n$ nonsingular

<u>Fact</u>: $u_1, \dots, u_n$ algebraically independent

the set of points $x \in X$ s.t. $u_1, \dots, u_n$ are local parameters. $\underline{du_1 \wedge \dots \wedge du_n (x) \neq 0}$ is

$\left(du_1, \dots, du_n \text{ linearly independent at } x \right)$

m_x / m_x^2

nonempty open subset.