tangent space : $x \in X \subset A^N$

$$x = 0, \quad X = \{F_1 = \cdots = F_k = 0\} \qquad F_i = L_i + F_i^{>1}$$

$$\| $$

$$\{ \ell : \ell \text{ is tangent to } X \text{ at } 0 \}$$

$$\|$$

$$\underline{\Theta_{x,X} = \{ L_1 = \cdots = L_k = 0 \} \text{ linear subspace of } A^N}$$

$$\|$$

$$(m_x / m_x^2)^*$$

$$\{ f \in k(X) : f \text{ is regular at } x \}$$

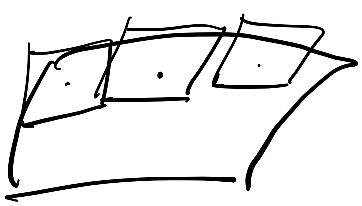
$$\|$$

$F \in m_x / m_x^2$ cotangent space , $m_x \subset \mathcal{O}_x \hookrightarrow k(X)$

maximal ideal

$$dF : \Theta_{x,X} \longrightarrow k$$
$$v \longmapsto dF(v).$$

m_x/m_x^2 is $\mathcal{O}_x/m_x = k$-vector.



$$A^N \times X \supset \Theta = \{ (a, x) : a \in \Theta_x \} \qquad \Theta_x$$

$$\downarrow \text{ regular surjective.} \qquad \downarrow$$

$$X \qquad\qquad\qquad x$$

X irreducible

$S = \min_{x \in X} \dim \Theta_x$. $\exists$ open subset $U \subset X$ s.t. $\dim \Theta_x = S$ $\forall x \in U$.

$x \in X$ nonsingular : $\dim \Theta_x = S$.

singular : $\dim \Theta_x > S$.

$\Leftarrow$ $\boxed{\text{Thm. on dim. of fibres}}$.

$\Downarrow$

Singular locus of X is a closed subset.

Thm: The dimension of the variety $\overset{\text{tr.deg.}\,k(X)}{=}$ $= \dim \Theta_X$ for nonsingular X.

$\qquad \overset{\|}{s}$

Pf: Fact: Every irreducible variety is birational to a hypersurface.

$$k(X) = k(z_1, \cdots, z_n, \underline{z_{n+1}}), \quad \underbrace{F(z_{n+1}; z_1, \cdots, z_n) = 0}_{\text{hypersurface}}$$

$$k(\underbrace{z_1, \cdots, z_n}_{\text{alg. ind.}}; z_{n+1}, \cdots, z_N)$$

(primitive element Thm.)

$$\begin{array}{ccc} X & \longrightarrow & Y \leftarrow \text{hypersurface} \\ \cup & & \cup \\ U & \xrightarrow{\cong} & V \end{array}$$

$\rightsquigarrow$ reduce to the hypersurface case

For hypersurface of dim n, $\sum_i \frac{\partial F}{\partial T_i}(x)\cdot(T_i - x_i) = 0$

$\mathbb{A}^{n+1} \supset \{\overset{\|}{F} = 0\}$

$\Rightarrow \quad s = n = \dim Y.$ ∎

· nonsingular point $x \in X$

$\dim_k m_x/m_x^2 = n \Rightarrow \underline{u_1, \cdots, u_n \in \mathcal{O}_x}$. $\{\bar{u}_1, \cdots, \bar{u}_n\}$ spans m_x/m_x^2 as vector space$/k$.

local parameters.

$\overset{\text{Nakayama}}{\Longrightarrow}$ $u_1, \cdots, u_n$ generate m_x

$\rightarrow$ power series expansions of $f \in \mathcal{O}_x$

$k[[T]]$ formal power series

$\Phi = F_0 + F_1 + F_2 + \cdots$ $F_i \in k[T]$ homogeneous of deg i.

is called Taylor series expansion of $f \in \mathcal{O}_x$

if $f - \underbrace{\sum_{i=1}^{k} F_i(u_1, \cdots, u_n)} \in m_x^{k+1}$

$x=0$

$F_0 = f(0)$. $f - f(0) \in m_x$ $\longrightarrow$ $\underline{f - F_0 - F_1} \in m_x^2$

$$f - f(0) = \sum a_i \underline{u_i} \mod m_x^2$$

$$f - F_0 - F_1 = \sum_j g_j h_j, \quad g_j, h_j \in m_x$$

$$g_j = \sum_j \beta_j u_j \mod m_x^2, \quad h_j = \sum \gamma_j u_j \mod m_x^2$$

$$f - F_0 - F_1 - \underbrace{\sum_{j,k} \beta_j \gamma_k u_j u_k} \in m_x^3$$

inductively $\longrightarrow$ Taylor expansion.

Thm: Every function $f \in \mathcal{O}_x$ has a unique Taylor expansion.

$$\Rightarrow \quad \mathcal{O}_x \hookrightarrow \underline{k[[T]]}.$$

$\mathcal{O}_x$ $\wr$ convergent power series

Thm: An irreducible subvariety $Y \subset X$ of codim 1 has a local equation in a nbhd. of any nonsingular point $x \in X$.

$\leftrightarrow$ global result: $\underset{\text{irred.}}{Y} \subset \mathbb{A}^n$ codim $Y = 1 \Rightarrow I_Y$ is principal.

$\exists$ nonzero polynomial F, $F=0$ on Y. $\{F=0\} \supset Y \Rightarrow \underline{\{F=0\} = Y}$
irreducible, codim 1

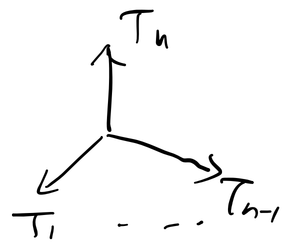
$G \in I_Y \underset{\text{Nullstellensatz}}{\Rightarrow} F \mid G^l \Rightarrow F \mid G \Rightarrow G \in (F)$ principal.

Fact: The local ring $\mathcal{O}_X$ of a nonsingular point is UFD.

$\leftsquigarrow \quad \hat{\mathcal{O}}_X = k[[T]]$ is UFD

Induction

$\leftsquigarrow$ Weierstrass Preparation thm



$\Phi \in k[[T]]$, $\Phi(0,\cdots,0,T_n) \neq 0$
$\quad\quad T_1,\cdots,T_n$

$\Rightarrow \quad \Phi \cdot U = T_n^m + R_1 T_n^{m-1} + \cdots + R_m. \quad R_i \in k[[T_1,\cdots,T_{n-1}]]$

(Unit $U(0,\cdots,0) \neq 0$)

Thm: If X is nonsingular. $\varphi: X \to \mathbb{P}^n$ rational map.

then the set of points at which φ is not regular has codim ≥ 2.

Pf: $\varphi = [f_0 : f_1 : \cdots : f_n]$ $\quad f_i \in k(X)$.

$f_0 = \dfrac{P_0}{Q_0}$, $\quad f_n = \dfrac{P_n}{Q_n}$ $\quad \rightsquigarrow f_i \in \mathcal{O}_X$ near x.

f_i has no common factor.

$Y \subset \{f_0 = \cdots = f_n = 0\}$

If codim $Y = 1$, then $I(Y) = (g)$, $\Rightarrow g \mid f_i$ $\quad \blacksquare$

Fact: Locus of indeterminacy of rational map to $\mathbb{P}^N$ has codim ≥ 2.

Cor: Any rational map of a nonsingular curve to $\mathbb{P}^N$ is regular.

Cor: If two <u>nonsingular</u> projective curve are birational, then they are isomorphic.

Thm: X affine var. $x \in X$ nonsingular point. $u_1, \cdots, u_n$ local parameters.

$\forall m \leq n$, $Y = \{u_1 = \cdots = u_m = 0\}$ is nonsingular at x

with $u_{m+1}, \cdots, u_n$ forming a system of parameters on Y at x.

Thm: $Y \subset X$ $x \in Y$ is nonsingular of both X and Y

$\rightsquigarrow$ $u_1, \cdots, u_n$ system of local parameters on X at x

s.t. $I(Y) = (u_1, \cdots, u_m)$.

- Normal variety

Def: An irreducible affine variety X is normal if $\underline{k[X]}$ is <u>integrally closed</u> in $\underline{k(X)}$.

$\forall f \in k(X)$ integral over $k[X]$ is already contained in $k[X]$.

Fact: X normal $\Rightarrow$ $\mathcal{O}_Y$ is integrally closed for any irreducible subvariety $Y \subset X$

$\mathcal{O}_Y = \{f : f \text{ is regular at some point of } Y\}$

X normal $\Leftarrow$ $\mathcal{O}_{x,X}$ is integrally closed for any $x \in X$

$k[X] \subset k(X)$

$\cup$ $\Rightarrow$ $f^n + a_1 f^{n-1} + \cdots + a_n = 0$.

integral f

$a_1 \in k[X] \hookrightarrow \mathcal{O}_x \hookrightarrow k(X)$

$\Rightarrow$ f is integral over $\mathcal{O}_x$ $\Rightarrow f \in \mathcal{O}_x$, $\forall x \in X$.

$$\Rightarrow f \in k[X] = \bigcap_{x \in X} \mathcal{O}_x .$$

Thm: A nonsingular variety is normal.

Pf: $\mathcal{O}_x$ is UFD $\Rightarrow$ $\mathcal{O}_x$ is integrally closed. ∎

Thm: If X affine is normal, $Y \subset X$ a codim 1 subvariety, then there exists an open affine $X' \subset X$ with $X' \cap Y \neq \phi$ s.t. the ideal of $Y' = X' \cap Y$ is $k[X']$ is principal.

Thm: The set of singular points of a normal variety has codim ≥ 2. (regular in codim. 1).

Cor: For algebraic curves, normal and nonsingular are equivalent conditions.

Proof: $S = \{\text{singular points of } X\} \subset X$

$\dim S < \dim X = n$

Suppose $\exists$ an irreducible component Y of S with $\dim Y = n-1$.

$\Rightarrow \exists\ X' \subset X$ affine open s.t. $Y' = X' \cap Y$ satisfies that $I(Y')$ is principal $= (g)$.

$\exists\ y \in Y'$ nonsingular on Y'

$\Rightarrow$ $\mathcal{O}_{Y',y}$ local ring. $u_1, \cdots, u_{n-1}$ local parameters on Y at y.

$$k[Y'] = k[X']/(g) \Rightarrow \mathcal{O}_{Y',y} = \mathcal{O}_{X',y}/(g).$$

$\mathcal{O}_{Y',y} \ni u_1, \cdots, u_{n-1} \leftarrow v_1, \cdots, v_{n-1} \in \mathcal{O}_{X',y}$

$u_1, \cdots, u_{n-1}$ generates $m_{Y',y}$ $\Downarrow$

$$n \geq \dim_k m_{X',y}/m_{X',y}^2 \Leftarrow v_1, \cdots, v_{n-1}, g \text{ generate } m_{X',y}$$

$\Downarrow$

$\dim_k m_{X',y}/m_{X',y}^2 = n \Rightarrow y$ is a nonsingular point on X.

$y \in Y = \{\text{singular points}\}$ contradiction

$\blacksquare$

$k(X) \supset k[X]$ $\qquad \overline{k[X]}$

$X \leftarrow X^{\nu}$ normalization of X

curve: $X \leftarrow X^{\nu}$ nonsingular curve