Thm on dimension of fibres:

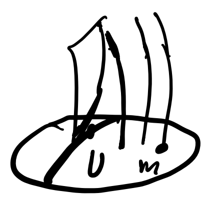
$f: X \to Y$ regular map between irred. varieties.

suppose f is surjective: $f(X) = Y$.

$\dim X = n$, $\dim Y = m$. Then

(i) $m \le n$, $\dim F \geq n-m$ for any $y \in Y$ and for any comp. F of $f^{-1}(y)$.

(ii) $\exists$ nonempty open subset $U \subset Y$ s.t. $\dim f^{-1}(y) = n-m$ $\forall y \in U$.



Pf: (i) Fact: $X \subset \mathbb{A}^N$ irreducible

F irred. polynomial on X

$\dim \underset{\{x : F(x) = 0\}}{\underset{\parallel}{X_F}} \geq n-1$. each component of X_F has $\dim \geq n-1$.

$\Rightarrow$ use m-times to get (i).

(ii) $W \subset Y$ open affine. $f: V \to W$ is dense.

$f^{-1}(W) \supset V \subset X$ open affine

$\Rightarrow$ $f^*: k[W] \hookrightarrow k[V]$,

$$k[W] = k[\underbrace{w_1, \cdots, w_m}_{\text{alg. ind.}}, w_{m+1}, \cdots, w_M]$$

$$k[V] = k[\underbrace{v_1, \cdots, v_n}_{\text{alg. ind.}}, v_{n+1}, \cdots, v_N].$$

$k(V)/k(W)$ has transcendental deg. $n-m$.

$v_1, \dots, v_{n-m}$ is alg. ind. over $k(W)$.

$v_i \in \underline{v_{n-m+1}, \dots, v_N}$ alg. dep. over $k(w_1, \dots, w_M, v_1, \dots, v_{n-m})$.

$\underline{F_i(\underbrace{v_i ; v_1, \dots, v_{n-m}}_{\text{polynomial in } v_i, v_1, \dots, v_{n-m}}; \underbrace{w_1, \dots, w_M}_{\text{coeff. fct.}}) = 0}$, $i = n-m+1, \dots, N$.

Need to prove $\dim f^{-1}(y) \leq n-m$ over some open subset.

nontrivial
$F_i(\bar{v}_i ; \bar{v}_1, \dots, \bar{v}_{n-m}) = 0$.

$\Rightarrow \bar{v}_i$ $\underline{\text{alg. dep. on } \bar{v}_1, \dots, \bar{v}_{n-m}}$

$\Rightarrow$ $\underline{\text{tr.deg. } f^{-1}(y) \leq n-m}$

Y_i = subvariety of Y given by the vanishing of leading term of F_i

$U = X \setminus (\bigcap_i Y_i) \ni y \Rightarrow \exists i$, s.t. $F_i(\bar{v}_i ; \bar{v}_1, \dots, \bar{v}_{n-m})$ has non-vanishing leading term.

$\blacksquare$

<u>Ex</u>: lines on surfaces in $\mathbb{P}^3$

$$\{\text{lines in } \mathbb{P}^3\} = Gr(2,4,\mathbb{C}) \hookrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$$

$$Gr(2,4,\mathbb{C}) = Gr(1,3,\mathbb{P}^3)$$

$$\{\text{lines in } \mathbb{P}^3\} \cong \{p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0\} = \Pi^4.$$

Fix degree of surface m, then

$$\{\text{surfaces of degree } m\} = \left\{\begin{array}{c}\text{hom. poly. of deg. } m \\ \text{in 4 variables}\end{array}\right\}/\sim$$

$$\mathbb{P}^N = \underline{\mathbb{P}^{\binom{m+3}{3}-1}} \qquad \binom{m+3}{3}$$

$$\mathbb{P}^N \times \Pi \supset \Gamma = \{(\xi, \eta) : L_\eta \subset S_\xi\}$$

$$\begin{tikzcd} & \Gamma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^N & & \Pi \end{tikzcd}$$

$\Pi \ni \eta \rightsquigarrow l$ $\qquad \mathbb{P}^3 = \{[u_0 : u_1 : u_2 : u_3]\}$

$$\{\text{surfaces that contain } l\} = \left\{u_0 G + u_1 H \;\; \begin{array}{l} G, H \text{ hom. poly.} \\ \text{of degree } m-1\end{array}\right\}/k^*$$

$$l = \underline{\{u_0 = u_1 = 0\}}$$

$$0 \to \ker \longrightarrow \underset{\substack{\| \\ \{\text{hom. poly. of deg. } m-1\}}}{V_{m-1}} \oplus V_{m-1} \longrightarrow V_m$$

$$(G, H) \longmapsto u_0 \cdot G + u_1 \cdot H$$

$$u_0 \cdot G + u_1 \cdot H = 0$$

$$\Downarrow$$

$$-\frac{G}{u_1} = \frac{H}{u_0} = H_1 \in V^{m-2}$$

$$2 \cdot \binom{m-1+3}{3} - \binom{m-2+3}{3}$$

$$= 2 \cdot \frac{(m+2)(m+1) \cdot m}{6} - \frac{(m+1) \cdot m \cdot (m-1)}{6}$$

$$= \frac{m(m+1)}{6}(2m+4-(m-1)) = \frac{1}{6} m(m+1)(m+5).$$

$$\Rightarrow \dim \Gamma = \underset{\dim \Pi}{\overset{\|}{4}} + \frac{1}{6} m(m+1)(m+5) - 1 = \frac{m(m+1)(m+5)}{6} + 3.$$

$P_1 : \Gamma \to \mathbb{P}^{N_m}$ $\qquad N_m = \binom{m+3}{3} - 1 = \frac{(m+3)(m+2)(m+1)}{6} - 1.$

$$\dim \Gamma - \dim \mathbb{P}^N = \frac{(m+1)}{6}\Big(\underbrace{m(m+5)}_{m^2+5m} - \underbrace{(m+3)(m+2)}_{m^2+5m+6} \Big) + 4$$

$$= -\frac{(m+1)}{\cancel{6}} \cancel{6} + 4 = \underline{3-m}.$$

$\Rightarrow$ · $m > 3$, P_1 is not surjective. $P_1(\Gamma)$ closed subset

$\Rightarrow \exists$ nonempty open subset of $\mathbb{P}^N$ s.t. surfaces corresponds to points in this open subset does not contain any line.

(most surfaces does not contain any line).

· $m=3$: · $\{T_1 T_2 T_3 = 1\} \subset A^3$

$\rightsquigarrow$ $X \subseteq \mathbb{P}^3$ has exactly 3 lines.

· $\overset{Z}{\overset{\shortparallel}{P_1(\Gamma)}} \subseteq \mathbb{P}^{N_3} = \mathbb{P}^{19}$ $\qquad \underset{\dim \Gamma}{\overset{\shortparallel}{N_3}} = \binom{3+3}{3} - 1 = \frac{6\times5\times4}{6} - 1$

closed subset.

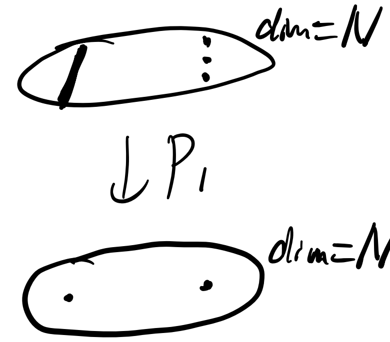
· $P_1 : \Gamma \to Z$ surjective.

If $Z \neq \mathbb{P}^{19}$, then $\dim Z \leq 18$.

$\Rightarrow$ $\dim P_1^{-1}(y) \geq \dim \Gamma - \dim Z = 19 - 18 = 1$

$\Rightarrow$ $Z = \mathbb{P}^{19}$ i.e. $P_1 : \Gamma \to \mathbb{P}^{19}$ is surjective.

$\Rightarrow$ · every cubic surface contains at least 1 line



· $\exists$ open subset of $\mathbb{P}^{19}$ whose points parametrize cubic surfaces containg containing finitely many lines.

(most cubic surfaces contain only finitely many lines)

<u>Fact</u>: Sm. cubic surface in $\mathbb{P}^3/\mathbb{C}$ has exactly 27 lines

· <u>Local properties</u>: · nonsingular/singular points.

tangent spaces, tangent cones.

· For nonsingular points: local parameters.

$\rightsquigarrow$ power series expansions of regular fct.

· local rings of non-singular points $\Leftrightarrow$ regular local ring.

· Normal varieties $\Longleftrightarrow$ integrally closed domains.

normalization of algebraic varieties.

local to global

$C(X) \subset \mathbb{C}^{N+1} \leftrightsquigarrow X \subset \mathbb{P}^N$

$C(X) = \{F_1 = \cdots = F_k\}$

$X = \{F_1 = \cdots = F_k = 0\}$

- X irreducible var. $\underline{x \in X}$ ← affine

$\mathcal{O}_x = \{f \in k(X) : f \text{ is regular at } x\} = k[X]_{m_x}$

$f = \dfrac{P}{Q}$ $\quad Q(x) \neq 0$.

m_x ← maximal ideal of $x \in X$.

A k-alg. prime ideal $P \subset A$

$A_P = \{(f, g) : g \notin P\}/\sim$ $\quad (f_1, g_1) \sim (f_2, g_2) \Leftrightarrow \exists h \notin P,\ h(f_1 g_2 - g_1 f_2) = 0$

Local ring of A at the prime ideal P.

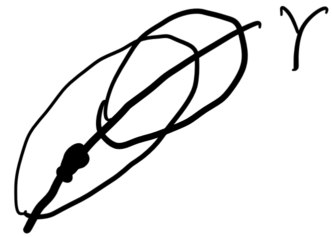
Y irreducible subvariety of $X \Longleftrightarrow P_Y$ prime ideal of $k[X]$

$P_Y = \{f \in k[X],\ f = 0 \text{ on } Y\}$

$\mathcal{O}_{X,Y} = \{f \in k(X) : \underline{f \text{ is regular at some point of } Y}\}$

$\mathcal{O}_{X,Y} = \left\{\dfrac{P}{Q} : Q \notin P_Y\right\}$

$\mathcal{O}_{X,Y} \supset m_Y = \{$ fct. that vanish on $Y \}$



$$\boxed{\mathcal{O}_{X,Y}/m_Y = k(Y)}$$

$\qquad\qquad\qquad\qquad \|$

$\{ f|_Y : f$ is regular at some point of $Y \}$

- X affine variety $\subset \mathbb{A}^N$, $x \in X$

Θ_x : tangent space at x $=$ $\{$ line ℓ that is tangent to X at $x \}$

$\|$

$\{F_1 = \cdots = F_k = 0\}$

assume $x = (0, \dots, 0) \in \mathbb{A}^N$

$\ell : \ x_i = a_i t$.

$\rightsquigarrow \mathrm{ord}_{t=0}(F_j(a_1 t, \dots, a_N t)) = k_j$

ℓ is tangent to X : $\quad \min\{k_j : j = 1, \dots, k\} \geq 2$.

$F_j = L_j + G_j$.

$\mathbb{A}^N$

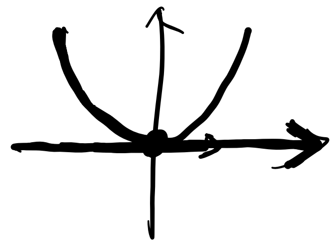
$\cup$

$\ell : \{x_i = a_i t\}$ tangent to $X \iff \{L_j(a) = 0 ; j = 1, \dots, k\}$

$\underline{Ex}$: $\overset{X}{\overset{\shortparallel}{\{F=0\}}} \subset A^N$, $0 \in X$.

tangent space : $\boxed{\{x : \sum_i \frac{\partial F}{\partial x_i}(0)\, x_i = 0\} \subset A^N}$

$\underline{y(y-x^2)=0.}$



tangent space at $0 \in X = A^2$

Intrinsic Definition of tangent space :

co-tangent space $=$ $\boxed{m_x / m_x^2 = \Theta_x^*}$

$dF : \Theta_x \longrightarrow k$

$x = (0, \cdots, 0)$. $F = F^{(1)} + F^{(2)} + \cdots$

$$d_x F : \underset{\underset{\Theta_x}{\cap}}{a} \mapsto F^{(1)}(a)$$

$$f: X \longrightarrow Y \quad \Rightarrow \quad f^*: k[Y] \to k[X]$$
$$x \longmapsto y \qquad\qquad f^*(m_y) \subset m_x$$
$$f^*(m_y^2) \subset m_x^2$$

$$\rightsquigarrow \quad f^*: m_y/m_y^2 \longrightarrow m_x/m_x^2$$

$$df: \Theta_x \longrightarrow \Theta_y. \quad \text{linear.}$$

- If f is isomorphic, then df is isomorphism.

- **Cor**: If $\dim \Theta_x = N$, then X can not be isomorphic to a subvariety of A^n for any $n < N$.

Pf: $f: X \longrightarrow Y \subset A^n$

$$df: \Theta_x \xrightarrow{\cong} \Theta_y \subset k^n \Rightarrow N \leq n.$$

$\dim m_x/m_x^2$ = embedded dim. of $x \in X$.