Dimension : X irred. quasi-proj. variety

$$\dim X = \text{tr.deg. } k(X)/k = n$$

$$k(X) = k(\underbrace{f_1, \cdots, f_n}_{\text{alg. indep.}}, \underbrace{f_{n+1}, \cdots, f_N}_{\text{alg. dep. on } f_1, \cdots, f_n})$$

- X irred. $\dim = n$. $Y \subset X$ closed subset
$\Rightarrow \dim Y \leq \dim X$. "$=$" iff $Y = X$

$$\boxed{\begin{array}{l} \cdot\ X^n \subset \mathbb{P}^N \text{ proj.} \quad \underline{F \text{ homog. poly. not } 0 \text{ on } X} \\ \qquad\qquad\qquad\qquad (\text{form}) \\ \quad X_F = \{F(x) = 0 ; x \in X\} \qquad \dim X_F = n-1. \end{array}}$$

Fact: Forms $F_0, \cdots, F_s$ forms of same degree m with no common zeros on X. Then the regular map

$$\begin{aligned} X &\to \mathbb{P}^s \\ x &\mapsto [F_0, \cdots, F_s] \end{aligned} \quad \text{is a finite map.}$$

- $\dim X = \max\{ n : \exists\, Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \supsetneq \phi \}$
Y_i: irred. sub. of X

$Y_0 = X,\ Y_1 = X_F,\ Y_2 = (Y_1')_{F_1}, \cdots$ (inductive def.)

Y_1' irred. comp. of Y_1

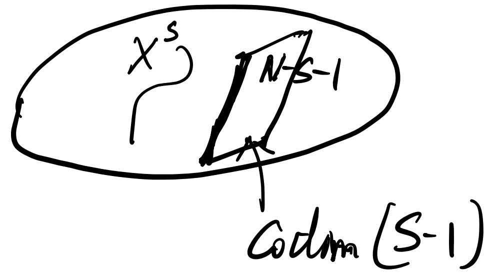
$X^n \subset \mathbb{P}^N$

$\dim X = \max \{ s : \exists$ linear subspace L of dim $N-s-1$ s.t. $L \cap X = \phi \}$

- L: $(N-n)$-dim subsp.

$\Rightarrow \dim(L \cap X) \geq 0 \Rightarrow L \cap X \neq \phi$.

$\Rightarrow \dim X \leq \max \{ \cdots \}$



- Find a linear subspace L of dim $N-n-1$ s.t. $X \cap L = \phi$.

$\dim \{ F_1 = \cdots = F_{n+1} = 0 \} \cap X = -1$

linear forms.

<u>Cor</u>: Any two curves of $\mathbb{P}^2$ intersect.

$\dim \{ \underbrace{F_1 = 0}_{1} \} \cap \{ F_2 = 0 \} \geq 0 \Rightarrow$ nonempty intersection

$\mathbb{P}^1 \times \mathbb{P}^1$

$(\mathbb{P}^1 \times \{0\}) \cap (\mathbb{P}^1 \times \{\infty\}) = \phi$

Cor (Tsen's Thm) $F(\underbrace{x_1, \dots, x_n})$ form in n variables of degree $m < n$ whose coefficients are polynomials in one variable t.

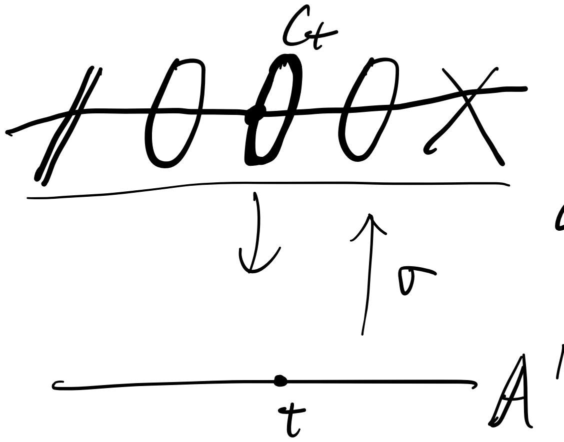
$$F(x_1, \dots, x_n) = \sum_{|\vec{m}| = m} \underbrace{a_{\vec{m}}(t)}\, x_1^{m_1} \cdots x_n^{m_n}$$

Then $\exists$ $x_i = P_i(t)$ polynomials, $F(P_1(t), \dots, P_n(t)) = 0$ as fcl. of t.

Ex: $n = 3$, $m = 2$.

$$X = \left\{ \sum_{i,j=0}^{2} a_{ij}(t) \cdot x_i x_j = 0 \right\} \subset \mathbb{P}^2 \times \mathbb{A}^1$$

$$\begin{array}{ccc} X & \ni & (t, [x_0, x_1 ; x_2]) \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & & t \end{array}$$



$\sigma : \mathbb{A}^1 \to X$

conic bundle

X a curve over $K = k(t)$. non-closed field.

Tsen's: X has a point with coordinates in K.

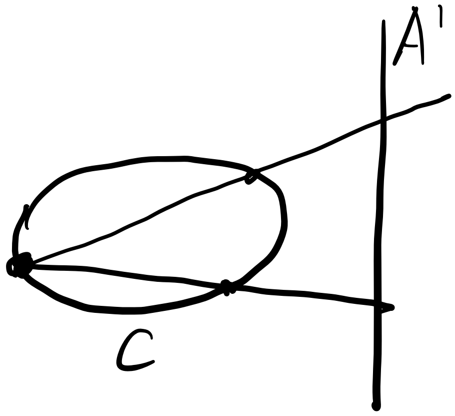
$x_i = P_i(t) \in k(t) = K$

If General point $t \in \mathbb{A}^1$, C_t is irreducible

Then X is non-degenerate over $K = k(t)$
conic

$$\Rightarrow \quad \underline{k(X) = K(X) \cong K(x) = k(t, x)} \Rightarrow X \text{ is rational}$$

$k(t,x) = k(\mathbb{A}^2)$



$\Rightarrow$ Cor: A nondegenerate conic bundle over $\mathbb{A}^1$ is a rational surface

Thm: X^n irred. proj. variety $\subset \mathbb{P}^N$

F: form $\not\equiv 0$ on X

$\Rightarrow X_F = \{x \in X : F(x) = 0\}$. Every component of X_F has dimension $n-1$.

Cor: $F_1, \cdots, F_m$ forms on X

$\Rightarrow$ Every irred comp. of $\{F_1 = \cdots = F_m = 0\} \cap X$

has dimension $\geq n - m$.

Thm: $X, Y \subset \mathbb{P}^N$ irred. quasi-proj. variety $\dim X = n$
$\dim Y = m$.

Then any component of $X \cap Y$ has $\underline{\dim. \geq n + m - N}$.

Moreover $n + m \geq N \Rightarrow X \cap Y \neq \phi$.

Pf: $X \cap Y \cong (\overset{n}{X} \times \overset{m}{Y}) \cap \Delta^{\leftarrow N}$

$$(X \times Y) \subset \mathbb{A}^N \times \mathbb{A}^N, \qquad \Delta = \{x_i - y_i = 0, i = 1, \cdots, N\}$$

$\dim (X \cap Y) \geq n + m - N$. ∎

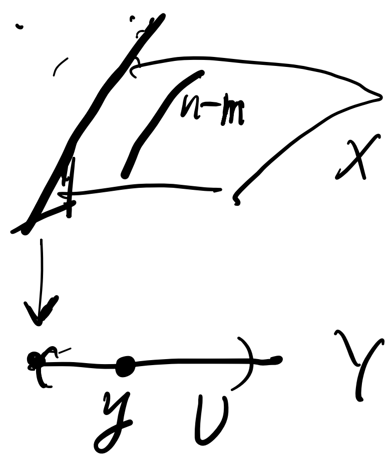
(Thm on the Dimension of Fibres)

Thm: $f: X \to Y$ regular map between irred. var. Suppose f is surjective. $\dim X = n$, $\dim Y = m$.

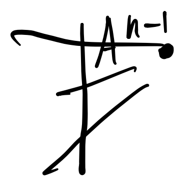
Then • $m \leq n$.

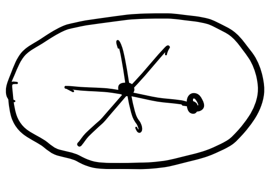
(i) $\dim F \geq n-m$, F is any component of $f^{-1}(y)$ for any $y \in Y$

(ii) there exists a non-empty subset $U \subset Y$ s.t. $\dim f^{-1}(y) = n-m$ for $y \in U$.



$\pi: Bl_0 \mathbb{A}^n \to \mathbb{A}^n$





Pf: $Y \subset \mathbb{A}^N$, $Z = \{F_1 = \cdots = F_m = 0\}$

$U \cap Z \cap Y = \{y\}$, $f^{-1}(y) = \{f^*F_1 = \cdots = f^*F_m = 0\}$ in $f^{-1}(U)$

$\Rightarrow$ Every compoent of $f^{-1}(y)$ has dim $\geq n-m$.

For (ii) Find open set $W \subset Y$
$V \subset f^{-1}(W)$

$f(V)$ dense in $W \Rightarrow f^*: k[W] \hookrightarrow k[V]$

$$\begin{array}{cc} k[W] & k[V] \\ \| & \| \\ k[w_1, \cdots, w_M] & k[v_1, \cdots, v_N] \end{array}$$

$k(V)$ has transcendent degree $n-m$ over $k(W)$

$v_1, \cdots, v_{n-m}$ alg. indep. over $k(W)$.

Construct $U \subset W$ s.t. $\forall y \in U$,

$$k[f^{-1}(y) \cap V] = k[\bar{v}_1, \cdots, \bar{v}_N]$$

$\bar{v}_1, \cdots, \bar{v}_{n-m}$ alg. independent $\Rightarrow \dim f^{-1}(y) = n-m$. ∎

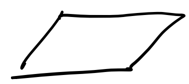
Lines on surfaces in $\mathbb{P}^3 = \{[x_0 : x_1 : x_2 : x_3]\}$

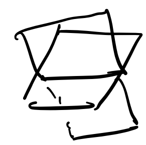
X surface in $\mathbb{P}^3 \Rightarrow X = \{F = 0\}$
homogeneous poly.

$\deg X = \deg X.$

- $\deg X = 1 \Rightarrow X = \text{plane} \cong \mathbb{P}^2$ infinite many lines.

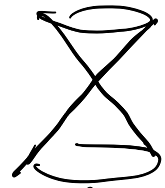
- $\deg X = 2 \Rightarrow F = \begin{cases} x_0^2 \\ x_0^2 + x_1^2 = (x_0 + \sqrt{-1}x_1)(x_0 - \sqrt{-1}x_1) \\ x_0^2 + x_1^2 + x_2^2 \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 \end{cases}$ $\rightarrow$





$\deg X = 2$
$\quad \|$
$\deg F$

infinitely many lines.



- $\deg X = 3 \Rightarrow$ · Every cubic surface has at least one line
  · most cubic surfaces has finitely many lines.
- $\deg X \geq 4 \Rightarrow \exists$ exist surfaces that does not have any lines

$\mathbb{P}^N = \{\text{surfaces of degree } m \text{ in } \mathbb{P}^3\}$

$= \underline{\{F : \text{degree } m \text{ hom. poly. in 4 variables}\}} / \sim$

$$N = \binom{m+3}{3} - 1$$

$$N = \binom{m+n}{n} = \dim \{\text{hom. poly. of deg } m \text{ in } n+1 \text{ var.}\}$$

$$\Pi = Gr(1, \mathbb{P}^3) = Gr(2, 4, \overset{2\cdot 2 = 4}{\mathbb{C}}) \hookrightarrow \mathbb{P}^5$$

$$\{P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0\}$$

$$Gr(k, n, \mathbb{C}) \to \mathbb{P}(\Lambda^k A^n)$$

$$\binom{n}{k} \quad \binom{4}{2} = 6$$

$$\mathbb{P}^N \times \Pi \supset \Gamma = \{(\xi, \eta) : \ell_\eta \subset X_\xi\}$$

$\xi \to$ surface X_ξ, $\eta \to \ell_\eta$

$$\begin{tikzcd} & \Gamma \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\ \mathbb{P}^N & & \eta \in \Pi \end{tikzcd}$$

$$X_\xi : u_0 \cdot \overset{m-1}{F} + u_1 \cdot \overset{m-1}{G} = 0$$

$$P_2^{-1}(\eta) = \{\xi \in \mathbb{P}^N : l_\eta \subset X_\xi\} \cong \mathbb{P}^{\frac{1}{6}m(m+1)(m+5)-1}$$

$$l_\eta = \{u_0 = u_1 = 0\}$$

$$\mathbb{P}^3 = \{[u_0, u_1, u_2, u_3]\}$$

$$\cong \mathbb{P}(\underline{u_0 \cdot F + u_1 \cdot G} : F, G \text{ degree } m-1 \text{ hom. po.})$$

$$(F, G) \xmapsto{\phi} u_0 \cdot F + u_1 \cdot G$$

$$\begin{array}{c}\cup\!\!\!\!\mid \\ V \oplus V \xrightarrow{\phi} W\end{array}$$

$$\dim W = \underbrace{\dim V \oplus V}_{\shortparallel\; 2 \cdot \binom{m-1+3}{3}} - \underbrace{\dim \mathrm{Ker}}_{\shortparallel\; \{(u_1 \cdot H, -u_0 \cdot H)\}} = \binom{m-2+3}{3}$$

H: $m-2$

$$= 2 \cdot \frac{1}{6} (m+2)(m+1) \cdot m - \frac{1}{6} \cdot (m+1) \cdot m (m-1)$$

$$= \frac{1}{6} \cdot m(m+1) \cdot (2m+4-m+1)$$

$$= \frac{1}{6} m(m+1)(m+5).$$