finite maps : $f: X \to Y$ regular map between affine.

Assume $f(X)$ is dense in $Y$ (in Zariski topology)

$f$ is a finite map if $f^*: k[Y] \hookrightarrow k[X]$ makes $k[X]$ a finite $k[Y]$-module i.e. $k[X]$ is integral over $k[Y]$: $\forall u \in k[X]$, $u^k + a_1 \cdot u^{k-1} + \cdots + a_{k-1} \cdot u + a_k = 0$ with $a_i \in k[Y]$.

$X, Y$ quasi-proj. $f: X \to Y$ is a finite map if $\forall y \in Y$, $\exists$ open nbhd $U$ of $y \in Y$ s.t. $f^{-1}(U)$ is affine and $f: f^{-1}(U) \to U$ is a finite map.

affine

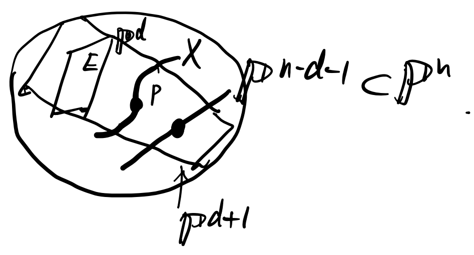
Fact: · $\forall y \in Y$, $f^{-1}(y)$ is finite subset of $X$ (quasi-finite)

· a finite map is surjective, and maps closed subsets to closed subsets.

Thm: $X \subset \mathbb{P}^n$ closed subvariety. $E \subset \mathbb{P}^n$ d dimensional linear subspace

$E \cap X = \phi$.

$\pi_E: X \to \mathbb{P}^{n-d-1}$ projection with center $E$ defines a finite map.



Pf: $y_0, \cdots, y_{n-d-1}$ homogeneous coordinates on $\mathbb{P}^{n-d-1}$

$\pi: X \to \mathbb{P}^{n-d-1}$ $\quad L_j$ linear forms. $E = \{L_j = 0\}$ $j=0,\dots,n-d-1$.

$\underset{[x_0:\cdots:x_n]}{x} \mapsto [\underset{y_j}{L_j(x)}, j=0,\dots,n-d-1]$

$$U_i = \pi^{-1}(\underset{\{y_i \neq 0\}}{A_i^{n-d-1}}) \cap X = \{x \in X : L_i(x) \neq 0\}$$

Need to prove: $\pi: U_i \to A_i^{n-d-1} \cap \pi(X)$ is a finite map.

$g\left(\frac{y_0}{y_i}, \cdots, \frac{\hat{y}_i}{y_i}, \cdots, \frac{y_{n-d-1}}{y_i}\right) = g \in k[U_i]$ is integral over $k[A_i^{n-d-1} \cap \pi(X)]$

$g = \dfrac{G_i(x_0, \dots, x_n)}{L_i^m}$ $\leftarrow$ form of deg. $m$.

Consider the map $\pi_1: X \to \mathbb{P}^{n-d}$ regular map

$x \mapsto [L_0^m, \cdots, L_{n-d-1}^m, G_i(x_0, \dots, x_n)] = [z_0 : \cdots : z_{n-d-1}, z_{n-d}]$

$\pi_1(X)$ is closed: $\pi_1(X) = \{F_1 = \cdots = F_s = 0\}$

$$\phi = \pi_1(X) \cap \{\underset{L_0^m}{z_0} = \cdots = \underset{L_{n-d-1}^m}{z_{n-d-1}} = 0\}$$

$$\Rightarrow \underset{k[z_0, \dots, z_{n-d}]}{(F_1, \dots, F_s, z_0, \dots, z_{n-d-1})} \supset \underset{z_{n-d}^k}{I_k} \text{ for some } k > 0$$

$$\Rightarrow \quad z_{n-d}^{k} = \sum_{j=0}^{n-d-1} z_j \cdot H_j + \sum_{j=1}^{s} F_j P_j$$

$$\Rightarrow \quad \underbrace{z_{n-d}^{k} = \sum_{j=0}^{n-d-1} z_j \cdot H_j^{(k-1)}} \quad \text{on } \pi_1(X).$$

$$\Rightarrow \quad z_{n-d}^{k} - \sum_{j=0}^{k-1} A_{k-j}(z_0, \dots, z_{n-d-1}) \cdot z_{n-d}^{j} = 0$$

$$\frac{G_i^k}{L_i^k} - \sum_{j=0}^{k-1} \frac{A_{k-j}(L_0^m, \dots, L_{n-d-1}^m) \cdot G_i^j}{L_i^k} = 0$$

$$\Rightarrow \quad g^k - \sum_{j=0}^{k-1} A_{k-j}(x_0^m, \dots, x_{n-d-1}^m) \cdot g^j = 0. \quad \blacksquare$$

<u>Thm</u>: $X \subset \mathbb{P}^n$, $F_0, \dots, F_s$ forms of degree $m$ having no common zeros on $X$. Then

$$\varphi(x) = (F_0(x) : \dots : F_s(x)).$$

defines a finite map.

<u>Pf</u>:

$$\begin{tikzcd} X \subset \mathbb{P}^n \arrow[r, hook, "\nu_m"] \arrow[dr, "\varphi"'] & \mathbb{P}^{\binom{n+m}{m}-1} \arrow[d, "\pi"] \\ & \mathbb{P}^s \end{tikzcd}$$

$$X \xrightarrow{\varphi} \mathbb{P}^s$$

$$\underline{\varphi^*(mH)} \cdot C = 0.$$

$$C \subset \varphi^{-1}(y).$$

$$\begin{tikzcd} \pi : \nu_m(X) \arrow[r] & \mathbb{P}^s \\ X \arrow[u, "\cong"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\Rightarrow$ $\pi$ finite $\Rightarrow$ $\varphi$ is finite

Thm: For any irred. proj. variety $X$, there ~~exists~~ a finite map $\varphi: X \to \mathbb{P}^m$.

$X \hookrightarrow \mathbb{P}^n$, $X \neq \mathbb{P}^n$, $x \in \mathbb{P}^n \setminus X$. $\pi_x: X \to \pi_x(X) \subset \mathbb{P}^{n-1}$ finite

If $\varphi(X) \neq \mathbb{P}^{n-1}$; $x_1 \in \mathbb{P}^{n-1} \setminus \varphi(X)$. $\pi_{x_1}: X \to \pi_{x_1}(X) \subset \mathbb{P}^{n-2}$

$\cdots \to \varphi: X \to \mathbb{P}^m$ finite and surjective.

Thm: For any irred. affine variety $X$, there exists a finite map $\varphi: X \to \mathbb{A}^m$. $\Rightarrow \dim X = m$. $\left( k(X) \text{ is finite alg. extension of } k(\mathbb{A}^m) \right)$

Def: Dimension of an irred. quasi-proj. variety $X$ is the transcendental degree of the function field $k(X)$

$X$ reducible. $\dim X = \text{maximum} \{ \dim X_i : X = \bigcup_i X_i,\ X_i \text{ irred. component} \}$

Fact: $X \subseteq Y$. Then $\dim X \leq \dim Y$.

If $Y$ is irred. $X \subseteq Y$ is closed subvariety then $\dim X = \dim Y$ iff $X = Y$.

Pf: Enough to prove when $X$ & $Y$ are irred. affine

$X \subset Y \subset \mathbb{A}^N$. $\dim Y = n$.

Then any $n+1$ coordinate functions are alg. dependent.

$\underline{F(t_{i_1}, \dots, t_{i_{n+1}}) = 0 \text{ on } Y \Rightarrow F = 0 \text{ on } X}$

$\Rightarrow$ trans.deg. $k(X)$ at most $n \Rightarrow \dim X \leq \dim Y$.

$$\begin{array}{c} k(X) \\ \shortparallel \\ k(t_1, \dots, t_N) \end{array}$$

Suppose $\dim X = \underline{\dim Y = n}$. $\Rightarrow$ $\underline{t_1, \dots, t_n \text{ alg. independent on } X}$.

$\Rightarrow \underline{t_1, \dots, t_n \text{ alg. ind. on } Y}$

$u \in k[Y]$, $u \neq 0$ on $Y \overset{?}{\Rightarrow} u \neq 0$ on $X$.

$(\Rightarrow I(Y) = I(X))$

$$a_0(t_1, \dots, t_n) u^k + \dots + \underbrace{a_k(t_1, \dots, t_n)}_{\neq 0} = 0 \text{ on } Y$$

$\Rightarrow$ true on $X$

If $u = 0 \Rightarrow a_k(t_1, \dots, t_n) = 0 \Rightarrow t_1, \dots, t_n$ alg. dep. on $X$

Contract to $t_1, \dots, t_n$ alg. indep. on $X$

$\Rightarrow u \neq 0$ on $X$. $\Rightarrow X = Y$.

<u>Thm</u>: <u>Every irred. comp. of a hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ has dimension $n-1$ (codim. 1).</u>

Pf: $X$ irred. comp. of hypersurface $\subset \mathbb{A}^n$. $\underline{\dim X \leq n-1.}$

$X = \{F = 0\}$ $\quad F(T_1, \dots, T_n)$

irred. polynomial $\quad F(0, \dots, 0, T_n) \neq 0.$

$\underline{G(t_1, \dots, t_{n-1}) = 0 \text{ on } X}. \Rightarrow F \mid G^l \Rightarrow F \mid G$ (no $T_n$)

$\Rightarrow G^l \in (F)$

$\Rightarrow$ $\underline{t_1, \dots, t_{n-1} \text{ alg. independent}}$ $\Rightarrow \dim X \geq n-1.$

$\Rightarrow \dim X = n-1.$

Thm: $X \subset \mathbb{A}^n$ variety. suppose all the components of $X$ have dimension $n-1$. Then $X$ is a hypersurface and the ideal $I(X)$ is principal.

Pf: Can assume that $X$ is irreducible. $X \neq \mathbb{A}^n$.

$\Rightarrow \exists \; F \equiv 0$ on $X$ ($F \neq 0$). $\Rightarrow \exists$ $\underline{\text{irreducible factor } H \text{ of } F}$ s.t. $H = 0$ on $X$

$Y = (H = 0)$. irreducible. $\quad X \subset Y$

$\dim X = \dim Y \Rightarrow X = Y = (H = 0).$

$G \in I(X). \Rightarrow G^l \in (H) \Leftrightarrow H \mid G^l \xRightarrow{H \text{ irred.}} H \mid G$

$\Rightarrow I(X) = (H)$ is principal.

In $\underline{\mathbb{P}^n \text{ or } \mathbb{A}^n}$, hypersurface $\Longleftrightarrow$ Codim 1 subvar.

$$\overset{||}{(F=0)}.$$


$X \cap (F=0)$  F degree m polynomial

**Thm**: $X \subset \mathbb{P}^N$ irred. proj. variety.

F : form (homogeneous polynomial) not zero on $X$

$X_F = \{x \in X : F(x) = 0\}$ has $\dim X_F = \dim X - 1$

$$\overset{||}{X} \cap \{\underbrace{F=0}_{n-1}\}$$ (intersection with hypersurface)

**Construction**: · Find $F(U_0, \cdots, U_N)$ form of any specified degree

s.t. F does not vanish on any components $X_i$ of $X$.

$\dim X_F < \dim X$. $X^{(0)}_{F_0} = X^{(1)} \supset X^{(1)}_{F_1} = X^{(2)} \supset X^{(2)}_{F_2} = X^{3} \supset \cdots$

$$\dim X^{(i)} > \dim X^{(i+1)}$$

$\dim X = n \Rightarrow \underline{X^{(n+1)} = \phi} \Rightarrow$ $F_0, F_1, \cdots, F_n$ do not common zero.

$X \xrightarrow{\varphi} \mathbb{P}^n$ is a finite map from $X$ to $\varphi(X)$

$x \mapsto [F_0 : F_1 : \cdots : F_{n-1} : F_n]$

$$\left.\begin{array}{l} \Rightarrow \dim X = \dim \varphi(X) = n \\ \quad \varphi(X) \subseteq \mathbb{P}^n \text{ closed} \end{array}\right\} \Rightarrow \varphi(X) = \mathbb{P}^n .$$

If $\underline{\dim X_{\underset{\substack{\shortparallel \\ F}}{F_0}} < n-1}$, then $X^{(n)}$ is empty

$\Rightarrow$ $F_0, \cdots, F_{n-1}$ does not common zero

$\Rightarrow$ $[0, \cdots, 0, 1] \notin \varphi(X)$ contradicts. $\varphi(X) = \mathbb{P}^n$.

$\Rightarrow$ $\dim X_F = n-1$.