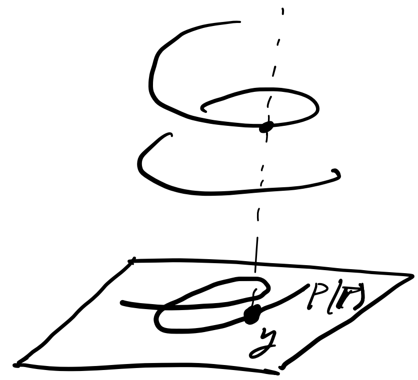
Thm: Image of projective variety under a regular map is closed.

Pf: · reduce to the graph case

If $\underset{\text{proj.}\ \ \ \text{quasi-proj.}}{\Gamma \subset X \times Y}$ is closed, then $P(\Gamma)$ is closed.

$P: X\times Y \to Y$

· reduce to $X \cong \mathbb{P}^n$, $Y \cong A^m$



· $\Gamma = \{ g_i(\underset{\text{homogeneous}}{u}; \overset{A^m}{y}) = 0 \}$

$P(\Gamma) = \{ y \in A^m$ s.t. $g_i(u;y) = 0,\ \forall i$ has a non-zero solution$\}$

$= \{ y \in A^m$ s.t. $\underbrace{(g_i(u;y))}_{\substack{\text{homogeneous} \\ \text{ideal}}} \not\supseteq I_s \quad \forall\, s = 1, 2, \cdots \}$ (Nullenstz)

$I_s = (\underbrace{u_0^{a_0} \cdots u_n^{a_n}}_{M^a} : a_0 + \cdots + a_n = s)$

$(g_i(u;y)) \supseteq I_s \iff M^a = \sum_i g_i(u,y) \cdot \underbrace{F_{i,a}(u)}$

homogeneous of degree $s - k_i$

$k_i = \deg_u g_i(u,y)$

$\iff \mathrm{Span}_{\mathbb{C}}(g_i(u,y) \cdot N_i^{\beta} ; i, \beta) = \left(\begin{array}{c} \text{hom. poly. in } u \\ \text{of degree } s \end{array} \right)$

N_i^β: homogeneous degree $s - k_i$

$g_i(u,y) \cdot N_i^\beta = \sum C^a_{i\beta} M^a$

$C^a_{i\beta}$: polynomial in y

$i\beta \to r$

$\Leftrightarrow \quad \operatorname{rank}(\underset{N\times M}{C_r^d}) = \dim \operatorname{Span}(M^d) = N$

$\Leftrightarrow \quad \underbrace{\exists\ N\times N \text{ submatrix whose determinant} \neq 0}$

$(g_i(u,y)) \not\supseteq I_s \Leftrightarrow \forall\ N\times N$ submatrix of (C_r^d), $\underbrace{\text{determinant} = 0}$.

$\Leftrightarrow p(T)$ is a closed subset of Y.

Cor: Regular maps from irreducible proj. var. to an affine variety must be constant.

- Finite maps :

X, Y affine varieties (isomorphic to closed subsets in A^n)

$f: X \to Y$ regular $\Longleftrightarrow$ $f^*: k[Y] \hookrightarrow k[X]$

Assume $f(X)$ is dense in Y $\qquad$ injective homomorphism of k-alg.

Def: f is a finite maps if $k[X]$ is integral over $k[Y]$.

i.e. $\forall\ u \in k[X]$, $u^k + b_1 u^{k-1} + \cdots + b_{k-1} u + b_k = 0$.

u is integral over $k[Y]$: $\qquad b_i \in k[Y]$

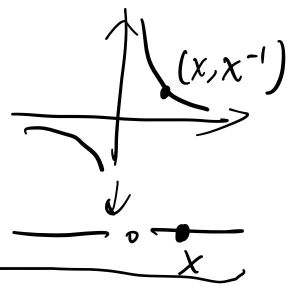
Prop: If f is a finite map, then $\forall y \in Y$ has at most a finite number of inverse images.

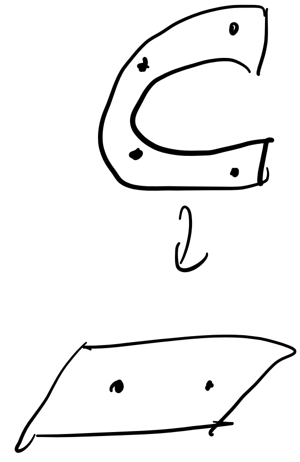
(quasi-finite)

Ex:

$$\begin{array}{ccc} X & & \\ \| & & \\ (x \cdot y = 1) & \subset & \mathbb{A}^2 \\ (x, y) & & \downarrow \\ \downarrow & & \\ x & & \mathbb{A}^1 = Y \end{array}$$

$y = x^{-1}$





$$k[X] = k[x, x^{-1}] \hookleftarrow k[Y] = k[x]$$

quasi-finite not finite

Proof:

$$X \subset \mathbb{A}^m \xrightarrow{f} Y \subset \mathbb{A}^n \qquad y \in Y$$

$$(x_1, \cdots, x_m) \qquad\qquad (y_1, \cdots, y_n)$$

$$x_i \in k[X] \xhookleftarrow{f^*} k[Y]$$

$$\underbrace{\left(x_i^k + a_1(y) x_i^{k-1} + \cdots + a_k(y) = 0 \right)}_{1 \le i \le m}$$ has finite many solutions

$\updownarrow$

$x \in f^{-1}(y)$. $\blacksquare$

Thm: A finite map is surjective.

Pf: $f: X \to Y$ finite map. $y=(a_1,\dots,a_n)\in Y$

affine

$m_y = (t_1-a_1,\dots,t_n-a_n)$

equations for $f^{-1}(y)$: $(f^*(t_1)-a_1,\dots,f^*(t_n)-a_n)$

$$f^{-1}(y)=\emptyset \iff (f^*(t_1)-a_1,\dots,f^*(t_n)-a_n)=k[X]$$

Hilbert's Nullstellensatz

$$(f^*(t_1)-a_1,\dots,f^*(t_n)-a_n) = m_y\cdot k[X]$$

$$\underset{\substack{\cup\!\!\mid \\ m_y}}{B = k[X]} \hookleftarrow \underset{\substack{\cup\!\!\mid \\ \mathfrak{a}}}{k[Y] = A}$$

$m_y = \mathfrak{a}$

B is integral

(i.e. B is a finite A-module)

$$\mathfrak{a}\cdot B = B \Rightarrow \mathfrak{a} = A$$

$$\Updownarrow$$

$$\boxed{\mathfrak{a}B \subsetneq B \Leftarrow \mathfrak{a} \subsetneq A}$$

Nakayama's Lemma: If M is a finite module over a ring A, $\mathfrak{a}\subset A$ an ideal. Suppose ($\forall x\in 1+\mathfrak{a}$, $x\cdot M=0$ implies $M=0$). Then $\mathfrak{a}\cdot M = M$ implies $M=0$.

$$\underbrace{\mathfrak{a} \subsetneq A \text{ then } 0 \notin 1+\mathfrak{a}} \qquad \underset{\not\ni\, 0}{x} \in 1+\mathfrak{a}\,, \quad \underset{\Downarrow\ x\cdot 1}{x\cdot B \neq 0}$$

$\Rightarrow$ ($\mathfrak{a}\cdot B = B$ implies $B=0$).

<u>Cor</u>: A finite map takes closed sets to closed sets.

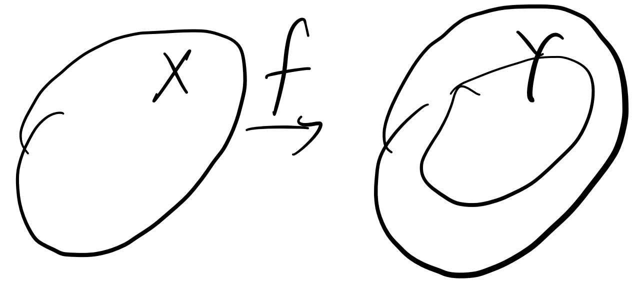
$$\begin{array}{ccc} f: X \to Y & & \bar{f}: Z \to \overline{f(Z)} \subset A^m \\ \cup & & \cap \\ Z & & Y \end{array}$$

$$\begin{array}{ccc} Z \to \overline{f(Z)} & & \\ \text{is surjective} & \Leftarrow & \left(\begin{array}{ccc} k[Z] \leftarrow k[\overline{f(Z)}] & \text{finite} \\ \| \qquad\qquad \| & \\ \underbrace{k[X]/I(Z)} \quad \underbrace{k[Y]/I(\overline{f(Z)})} & \end{array} \right) \\ \Updownarrow & & \\ f(Z) = \overline{f(Z)} & & \end{array}$$

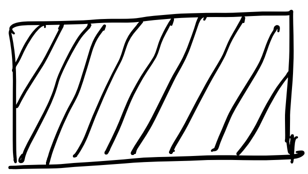
<u>Def</u>: $f: X \to Y$ regular map of quasi-projective var. is finite if $\forall y \in Y$, $\exists$ affine nbhd. V of y s.t. $U = f^{-1}(V)$ is affine and $f: U \to V$ is finite.

<u>Fact</u>: Def. does not depend on the choice of such V.

Thm: If $f: X \to Y$ is a regular map and $f(X)$ is dense in Y, then $f(X)$ contains an open set of Y.



Ex: $f: \mathbb{R}^1 \to T$



$f: X \to Y$ dense image $\rightsquigarrow$ $f^*: k[Y] \hookrightarrow k[X]$

$$k(Y) \hookrightarrow k(X)$$

$$\dim Y = \text{tr.deg.}\, k(Y) \leq \text{tr.deg.}\, k(X) = \dim X$$

Pf: Reduce to case: X, Y are both irreducible and affine.

$k[Y] \hookrightarrow k[X]$ Choose r element $u_1, \dots, u_r \in k[X]$

$k(Y) \hookrightarrow k(X)$ that are algebraically independent

$r = \text{tr.deg.}(k(X)/k(Y))$ over $k[Y]$.

$$k[X] \supset \underbrace{k[Y][u_1, \cdots, u_r]}_{} \supset k[Y]$$

$\Downarrow$ algebraic

$$k[Y][u_1,\cdots,u_r] = k[Y \times A^r]$$

$$\boxed{X \xrightarrow{h} Y \times A^r \xrightarrow{g} Y}$$

f

$k[X] = k[v_1, \cdots, v_m]$ s.t. $a_i v_i$ is integral over $k[Y \times A^r]$

$k[Y \times A^r] \ni a_1, \cdots, a_m$

$\boxed{F = a_1 \cdots a_m}$. $D(F) = \{F \neq 0\} \subset Y \times A^r$

$\Rightarrow$ v_i is integral on $\underline{D(h^* F) \subset X}$ over

$k[Y \times A^r][1/F]$

$$\frac{(a_i v_i)^k + b_1 (a_i v_i)^{k-1} + \cdots + b_{k-1}(a_i v_i) + b_k = 0}{a_i^k}$$

$$v_i^k + (b_1 \cdot (a_i^{-1})) v_i^{k-1} + \cdots + (b_{k-1} a_i^{-(k-1)}) v_i + (b_k \cdot a_i^{-k}) = 0$$

$$a_i^{-s} = F^{-s}(a_1^s \cdots \hat{a}_i^s \cdots a_m^s) \quad k[Y \times A^r][\tfrac{1}{F}]$$

$$\boxed{h: D(h^*(F)) \to D(F) \text{ is finite.}}$$

$$\Rightarrow \boxed{h(D(h^*F)) = D(F) \subset h(X)}$$

It remains to prove that $g(\underline{D(F)})$ contains an open subset. $F \in k[Y][u_1, \dots, u_r]$

$$\boxed{F = \sum_{\alpha} F_\alpha(y) \cdot T^\alpha}$$

$(\underline{F(y,c) \neq 0} \Leftarrow \exists \alpha, \underline{F_\alpha(y) \neq 0}) \Rightarrow$

$\exists c$ s.t.

$$\underline{g((F \neq 0)) \supset \bigcup D(F_\alpha)}$$

$$X \xrightarrow{h} Y \times \mathbb{A}^r$$

$D(h^*F)$ finite $\longrightarrow$

$\downarrow$

Y

$D(F)$

$\bigcup_\alpha D(F_\alpha)$.