X: quasi-projective variety = open subset of $\underline{\text{closed projective set}}$.

$$\parallel$$

$$\mathbb{P}^N \supset \{F_1 = \cdots = F_k = 0\}$$

rational function: $f = \frac{P}{Q}$ P, Q homogeneous same degree

$$\frac{P_1}{Q_1} = \frac{P_2}{Q_2} \Leftrightarrow \underline{P_1 Q_2 - P_2 Q_1 = 0 \text{ on } X}$$

$$(F(\lambda T) = \lambda^{\deg F} F(T))$$

f is regular at $x \in X$: $\exists$ rep. $f = \frac{P}{Q}$ s.t. $Q(x) \neq 0$.

$k[X] = \{f \in k(X) : f \text{ is regular at every } x \in X\}$.

- If X is closed subset of A^n, then $k[X] = k[T]/I_X$ f.g. k-alg.
- If X is irred. closed proj. set, $k[X] = k$.
- In general, $k[X]$ can be non f.g. [Rees, Nagata)

Def: $\underline{\text{Affine variety}}$: quasi-proj. var. that is $\underline{\text{isomorphic}}$ to a closed subset of A^N

Ex: $x \in \underline{X = A^1 \setminus 0}$. not closed subset of A^1 is an $\underline{\text{affine variety}}$.

$$\begin{array}{c} \{x \neq 0\} \\ \parallel \\ X = A^1 \setminus 0 \\ \uparrow \quad \wr\wr \\ (x,y) \quad \underline{\{x \cdot y = 1\} \subset A^2} \end{array}$$

Proj. variety: quasi-proj. var. that is isomorphic to a closed proj. subset.

Lem: The property that a subset $Y \subset X$ is a closed in a quasi-proj. variety is a local property.

This means: if $X = \bigcup_{\alpha} U_\alpha$ and $Y \cap U_\alpha$ is closed in U_α

Then Y is closed in X.

Lem: Every $x \in X$ (quasi-proj) has nbhd. that is isomorphic to an affine variety.

Pf: $x \in X \subset \mathbb{P}^n$. $x \in A^n_0 = \{[u_0 : u_1 : \cdots : u_n] : u_0 \neq 0\}$

$x \in \underbrace{X \cap A^n_0} = Y \setminus Y_1$ $x \notin Y_1$ closed subset of A^n.

$\Rightarrow$ can Find $F \in k[T]$, s.t. $F = 0$ on Y_1, $F(x) \neq 0$

$D(F) = Y \setminus Z(F)$ open subset containing x

Y is closed $\Rightarrow$ $Y = \{G_1 = \cdots = G_m = 0\}$ $G_i \in k[T]$.

$W \subset A^{n+1}$: $\begin{cases} G_1(T) = \cdots = G_m(T) = 0 \\ F(T) \cdot U = 1 \end{cases}$, $G_i, F \cdot U \in k[T, U]$ $\quad T = T_1, \cdots, T_n$

$\text{closed subset} \nearrow W \subset A^{n+1} \downarrow A^n$

$W \simeq D(F) = \{F \neq 0\}$ $\Rightarrow$ $D(F)$ is an affine variety.

$(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_n)$

$(y_1, \cdots, y_n, \frac{1}{F(y)}) \leftarrow (y_1, \cdots, y_n)$

$\blacksquare$

· $f: X \dashrightarrow Y \subset \mathbb{P}^m$ rational map

$$f = [F_0 : \cdots : F_m] \qquad F_i \text{ same degree homogeneous}$$

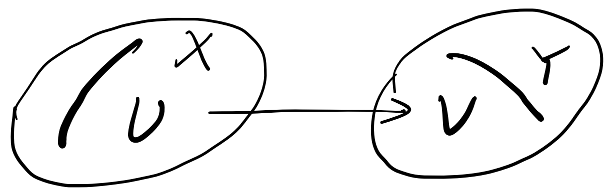
$$[F_0 : \cdots : F_m] = [G_0 : \cdots : G_m] \quad \text{if } F_i \cdot G_j - F_j \cdot G_i = 0 \text{ on } X$$

<u>**Prop.**</u> Two irred. var. X and Y birational iff

$$\left(\begin{array}{ll} f: X \to Y & \text{s.t. } f(X) \text{ is dense in } Y \\ g: Y \to X & f \circ g = 1 \text{ on domain of} \\ & g \circ f = 1 \quad \text{definition} \end{array} \right)$$

they contain isomorphic open subsets $U \subset X$ and $V \subset Y$.

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & Y \\ \cap & & \cup \\ X & \xleftarrow{g} & V_1 \end{array}$$



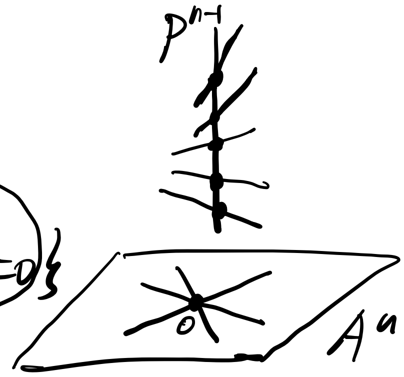
$U = f^{-1}(V_1) \cap U_1$, $V = g^{-1}(U_1) \cap V_1$, open subsets

$U \subset X$, $V \subset Y$

$f(U) = V$, $g(V) = U$. $f \circ g = 1$ on V

$g \circ f = 1$ on U .

Ex: Blowup $0 \in \mathbb{A}^n$



$$X = Bl_0 k^n = \{ (\underset{(x_1,\cdots,x_n)}{x}, \underset{[u_0:\cdots:u_n]}{[u]}) : x_i u_j - x_j u_i = 0 \ \ \forall i,j \} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

$\pi : X \to \mathbb{A}^n$. $(x : [u]) \longmapsto x$.

$\pi^{-1} : \mathbb{A}^n \dashrightarrow X$ domain of definition $= \mathbb{A}^n \setminus 0$.

$\pi^{-1}(0) = P^{n-1}$ $\blacksquare$

- Products of quasi-projective variety

$$\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{f} \mathbb{P}^{(m+1)(n+1)-1} \quad \text{Segre embedding}$$

$$([u_0,\cdots,u_m], [v_0,\cdots,v_n]) \longmapsto [\underbrace{u_i \cdot v_j}_{w_{ij}}] \quad \begin{matrix} 0 \le i \le m \\ 0 \le j \le n \end{matrix}$$

$$\operatorname{Im} f = \left\{ w_{ij} w_{kl} - w_{il} w_{jk} = 0 : \begin{matrix} 0 \le i,j \le m \\ 0 \le k,l \le n \end{matrix} \right\}$$

closed alg. subvariety of $\mathbb{P}^m \times \mathbb{P}^n$:

$$F_k(\underbrace{u_i v_j}_{w_{ij}}) = 0, \qquad k = 1, \cdots, r.$$

homogeneous in w_{ij}

$G_k(u_0,\cdots,u_m; v_0,\cdots,v_n)$ bihomogeneous in u and v same degree

$\underline{G(u_0, \dots, u_m; v_0, \dots, v_n) \text{ bihomogeneous}}$ in $\underline{u \text{ of deg } r.}$
$\underline{v \text{ of deg } s.}$

if $r > s$, then $\underline{G(u_0, \dots, u_m; v_0, \dots, v_n) = 0}$

$$\Updownarrow$$

$$\{ v_j^{r-s} \, \underline{G(u_0, \dots, u_m; v_0, \dots, v_n) = 0}, \ j = 0, \dots, n. \}$$

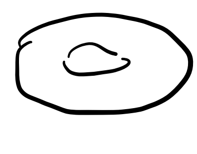
closed subset of $\underline{P^n \times A^m}$: $g_k(\underbrace{u_0 : \dots : u_n}_{\text{homogeneous}}; \underbrace{y_1, \dots, y_m}_{A^m}) = 0.$

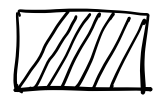
homogeneous in u

$X = \underline{X_1} \setminus \underline{X_0}$, $Y = \underline{Y_1} \setminus \underline{Y_0}$ quasi-projective

$X \times Y = (X_1 \times Y_1) \setminus ((X_1 \times Y_0) \cup (X_0 \times Y_1))$ is quasi-projective.

<u>Thm</u>: The image of a projective variety under a regular map is closed.

not true for smooth maps: $\mathbb{R} \to$ 



<u>Cor</u>: Regular functions on an <u>irreducible</u> projective variety must be constant

<u>Pf</u>: $f: X \to A^1$. $f(X)$ closed $\Rightarrow$ $f(X)$ finite $= \{a_1, \dots, a_r\}$

$$X = \bigcup_i \underline{f^{-1}(a_i)}$$

$\xRightarrow{X \text{ irred.}}$ $f(X) = \{a\}$ i.e. f is constant.

Cor: A regular map $f: X \to Y$ from irreducible proj. var. X to an affine variety Y maps X to a point.

Pf: $Y \subset \mathbb{A}^m$, $f = (\varphi_1(x), \dots, \varphi_m(x))$, φ_i regular function
$\Rightarrow \varphi_i$ constant.

Pf of the thm: · The graph of $f = \overset{\Gamma_f}{\shortparallel}$ $\{(x, f(x)) \in X \times Y\}$ is closed subset of $X \times Y$.

X: projective, $Y \hookrightarrow \mathbb{P}^m$: quasi-proj.

$$\Phi = (f, \mathrm{id}): X \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m$$
$$(x, y) \mapsto \underline{(f(x), y)}$$

$$\Gamma_f = \Phi^{-1}(\Delta), \quad \underbrace{\Delta = \{(y, y) : y \in \overset{[u_0 : \dots : u_m]}{\mathbb{P}^m}\}}_{= \{u_i v_j - u_j v_i = 0\}} \subset \mathbb{P}^m \times \mathbb{P}^m$$

Fact: Preimages of closed subsets under regular maps are also closed. (by pulling back of defining fct.

$$\begin{tikzcd} & X \times Y \arrow[d, "p"] \\ X \arrow[ur] \arrow[r, "f"] & Y \end{tikzcd}$$
$x \mapsto (x, f(x))$

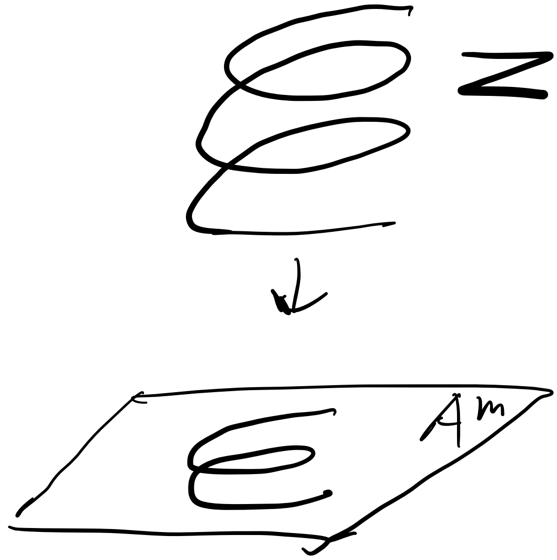
$$f(X) = p(\Gamma_f)$$

Need to prove: X proj. ($X \hookrightarrow \mathbb{P}^n$), Y quasi-proj. ($Y = \bigcup_\alpha U_\alpha$)

$p: X \times Y \to Y$ maps closed subsets to closed subsets.

Just need to prove this for the case $X = \mathbb{P}^n$, $Y = \mathbb{A}^m$

$$g_k(\underbrace{u_0, \cdots, u_n}\,;\, \underbrace{y_1, \cdots, y_m})$$



$$p(Z) = \{ y \in \mathbb{A}^m : p^{-1}(y) \neq \phi \}$$

$$= \{ y \in \mathbb{A}^m : \underbrace{(g_k(u; y))_k \text{ has a}}_{\text{nonzero solution}} \}$$

Nullstellensatz

$$= \{ y \in \mathbb{A}^m : \underline{(g_1(u;y), \cdots, g_t(u;y)) \not\supset I_s} \text{ for all } s = 1, 2, \cdots \}$$

$$\boxed{\begin{array}{l} \text{For each } s \in \{1, 2, \cdots\}, \\ \{ y \in \mathbb{A}^m : \underline{(g_1(u;y), \cdots, g_t(u;y)) \not\supset I_s} \} \\ \text{is closed subset of } \mathbb{P}^n \end{array}}$$

$$I_s = (u_0, \cdots, u_n)^s = (\underbrace{u_0^{\alpha_0} \cdots u_n^{\alpha_n}}_{M^\alpha} : \alpha_0 + \cdots + \alpha_n = s)$$

$$\underline{(g_1(u;y), \cdots, g_t(u;y)) \supset I_s}$$

$$\iff \underbrace{M^\alpha}_{s} = \sum \underbrace{g_i(u,y)}_{k_i} \cdot \underbrace{F_{i,\alpha}(u)}_{s-k_i}, \quad \forall \alpha = (\alpha_0, \cdots, \alpha_n) \text{ s.t. } \alpha_0 + \cdots + \alpha_n = s,$$

$$\iff \underbrace{\operatorname{Span}\{ \underline{g_i(u,y) \cdot N_i^\beta} \}}_{V_1} = \underbrace{\{ \text{homogeneous poly. in } u \text{ of degree } s \}}_{V_2}$$

N_i^β: homog. monomial in u of degree $s - k_i$

$$(g_1(u,y),\dots,g_t(u,y)) \not\supset I_s$$

$$\iff \mathrm{Span}\{v_1,\dots,v_r\} \subsetneq \{u_1,\dots,u_q\}$$

$$\underline{\overset{\parallel}{V_1} \subsetneq V_2}$$

$$v_i = a_{il} u_l \qquad \boxed{\mathrm{rk}\begin{pmatrix} \quad \end{pmatrix} < \overset{N}{\overset{\parallel}{\dim V_2}}}$$

$$\Updownarrow$$

closed condition. $\Leftarrow$ $N \times N$ minor has determinant 0.