Rational Map: $$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cap & & \cap \\ A^n & & A^m \end{array}$$

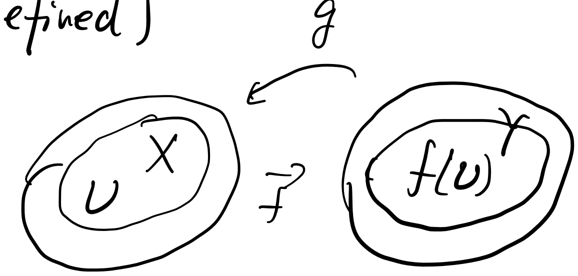
$$f = (\varphi_1, \dots, \varphi_m) \qquad \varphi_i = \frac{P_i}{Q_i}$$

Domain of definition $= \{x : f \text{ is regular at } x\} \subset X$

- Assume $\underline{f(X) \subset Y \text{ is dense}}$. Then
$$f^* : k(Y) \hookrightarrow k(X) \quad \text{injective}$$

- $f: X \to Y$ birational if $f(X)$ is dense in Y
$$g: Y \to X \quad \text{rational}$$
s.t. $f \circ g = g \circ f = 1$ (where defined)

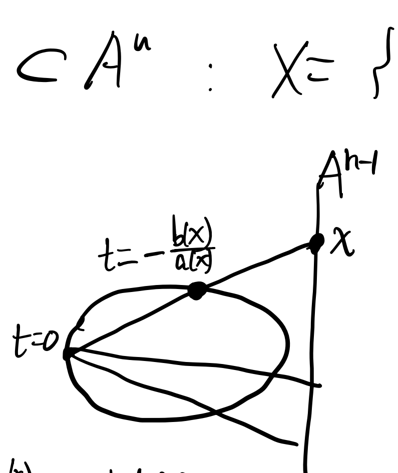
<u>X is called birational to Y</u>



<u>X is Rational</u>: X is birational to A^n.

<u>Ex</u>: Irreducible quadric $\subset A^n$: $X = \{F(T_1, \dots, T_n) = 0\}$

Fix $p \in X$
$\quad \| $
$(0, \dots, 0)$

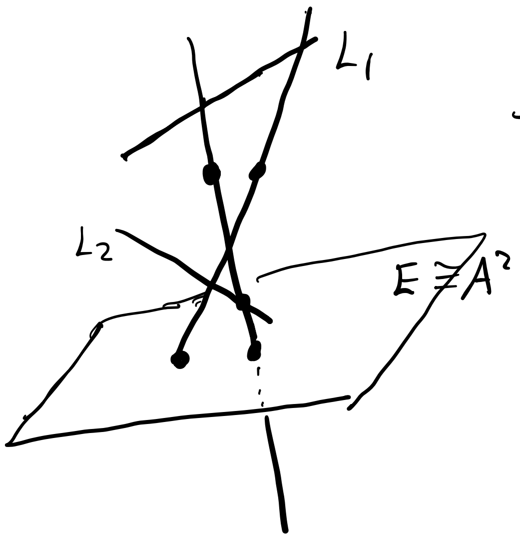


$$T_1 = x_1 \cdot t$$
$$\vdots$$
$$T_{n-1} = x_{n-1} \cdot t$$
$$T_n = t$$
$$\Downarrow$$
$$F(x_1 t, \dots, x_{n-1} t, t)$$
$$\|$$
$$a \cdot t^2 + bt = 0$$
$$\Rightarrow t = 0, \ -\frac{b(x)}{a(x)}$$

$$(x_1, \dots x_{n-1}) \to \left(-x_1 \frac{b(x)}{a(x)}, \dots, -x_{n-1} \frac{b(x)}{a(x)}, -\frac{b(x)}{a(x)}\right)$$
$$\qquad \qquad \qquad \qquad \cap$$
$$\qquad \qquad \qquad \qquad A^n$$

Ex: $\overbrace{\{x^3+y^3+z^3=1\}}^{X} \subset A^3$

L_1: $x+y=0, z=1$; L_2: $x+\varepsilon y=0$, $z=\varepsilon$, $\varepsilon^3=1$, $\varepsilon \neq 1$



$f: X \to A^2$ birational

Question: Are $\overset{\text{smooth}}{\vee}$ cubic hypersurfaces rational?

dim 1: ✗

dim 2: ✓

dim 3: ✗

dim 4: In general not rational.

Fact: Any irred. closed set X is birational to a hypersurface of some affine space A^m

- closed subset of projective space

$X \subset \mathbb{P}^n$ $\quad X = \{F_1 = \cdots = F_k = 0\}$

$F \in k[S_0, \dots, S_n]$. $\quad P \in \mathbb{P}^n$, $P = a = [a_0, \dots, a_n] = [\lambda a_0, \dots, \lambda a_n]$

$F = f_0 + f_1 + \cdots + f_r$

$\forall \lambda \in k$

$F(a) = 0$, $0 = F(\lambda a) = f_0(a) + \lambda f_1(a) + \lambda^2 f_2(a) + \cdots + \lambda^r f_r(a)$

$\Rightarrow f_0(a) = f_1(a) = \cdots = f_r(a) = 0$.

$I(X) = \{ f \text{ homogeneous} : f \text{ vanishes at } x \in X \}$.

homogeneous ideal

$= (f_1, \cdots, f_k)$

Ex: $F \in k[S_0, \cdots, S_n]_d$, $\{F=0\} = X$.

$\lambda^d F(\partial) = F(\lambda \partial)$

Ex: Grassmannian: V vector space. $V \simeq k^N$.

$$L^r \in \{ r\text{-dim linear subspace } L^r \subset V \} = \underbrace{Gr(r, N, k)}_{r(N-r)}$$

$$\begin{aligned} &\parallel \\ &\mathrm{Span}\{\underbrace{f_1, \cdots, f_r}_{\text{a basis}}\} \mapsto f_1 \wedge f_2 \wedge \cdots \wedge f_r \in \underbrace{\Lambda^r V}_{\binom{N}{r}} \\ &\qquad\qquad \uparrow T \\ &f_1', \cdots, f_r' \qquad f_1' \wedge \cdots \wedge f_r' = \overset{\neq 0}{\det(T)} f_1 \wedge \cdots \wedge f_r \end{aligned}$$

$\rightsquigarrow [f_1 \wedge f_2 \wedge \cdots \wedge f_r] \in \underline{\mathbb{P}(\Lambda^r V) \cong \mathbb{P}^{\binom{N}{r}-1}}$

$Gr \hookrightarrow$

$\underline{\{e_1, \cdots, e_N\} \text{ basis for } V}$

$$f_1 \wedge \cdots \wedge f_r = \sum_{1 \le i_1 < \cdots < i_r \le N} P_{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \underset{1 \le i_1 < \cdots < i_r \le N}{[P_{i_1 \cdots i_r}]}$$

$$F = (f_1 \cdots f_r) = \overset{E}{\overset{\parallel}{(e_1 \cdots e_N)}} \cdot A_{N \times r}$$

$$F \cdot T = E(AT) \qquad AT = \begin{array}{c} \overbrace{\qquad}^{r} \\ \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline * & \cdots & * \\ * & \cdots & * \end{array}\right) \end{array} \begin{array}{l} \}\, r \\ \\ \}\, N-r \end{array}$$

$x \in \underline{\Lambda^r V} = \mathrm{Span}\{e_{i_1}\wedge\cdots\wedge e_{i_r}\}$

x is decomposable : $\boxed{x = f_1\wedge\cdots\wedge f_r} \rightsquigarrow L = \mathrm{Span}\{f_1,\cdots,f_r\}$

$$\Longleftrightarrow \quad \underline{(\underset{V}{y \lrcorner x}) \wedge \underset{\Lambda^r V}{x} = 0}, \quad \forall\ \underline{y \in \Lambda^{r-1}V^*}$$

(plücker relation)

$x = \sum P_{\underbrace{i_1\cdots i_r}} e_{i_1}\wedge\cdots\wedge e_{i_r}$, $\quad y = u_{j_1}\wedge\cdots\wedge u_{j_{r-1}}$

$\underset{V^*}{\{u_j\}} \overset{\text{dual}}{\Longleftrightarrow} \{e_i\}$

$$\sum_{t=1}^{r+1}(-1)^t P_{\underbrace{i_1\cdots i_{r-1}}j_t} P_{\underbrace{j_1\cdots\hat{j}_t\cdots j_{r+1}}} = 0$$

(Quadratic equations)

$G \hookrightarrow \mathbb{P}(\Lambda^r V)$

$\underline{r=2}$: $\forall y \in \Lambda^{2-1}V^* = V^*$. $(y \lrcorner x)\wedge x = \frac{1}{2}\, y \lrcorner (x\wedge x) = 0$

$\Longleftrightarrow \boxed{\underset{\Lambda^4 V}{x\wedge x} = 0}$ $\qquad x = \frac{1}{2}\sum P_{ij}\, e_i \wedge e_j$

$n=4 \quad \underset{\text{hypersurface}}{\underbrace{\{P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0\}}} \subset \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$

$\underline{Gr(2,4)}$

Lem: $I \subset k[S]$ homogeneous ideal

$Z(I)$ is empty iff $\underline{I \text{ contains } I_s \text{ for } s > 0}$.

$$I_s = \boxed{(S_0, \cdots, S_n)^s} = (S^m : |m| = s), \quad |m| = m_0 + \cdots + m_n, \quad S^m = S_0^{m_0} \cdots S_n^{m_n}$$

pf: "$\Leftarrow$" $S_0^s = \cdots = S_n^s = 0$

$\left(\mathbb{P}^n = (k^{n+1} - \{0\}) / k^*\right)$ $\Rightarrow \underline{S_0 = \cdots = S_n = 0}$

$\Rightarrow Z(I) = \phi$.

"$\Rightarrow$" homogeneous basis of I: $\underline{\{F_1, \cdots, F_r\}}$

$(F_i(1, T_1, \cdots, T_n) = 0) = \phi$, $\quad T_1 = \frac{S_1}{S_0}, \ldots, T_n = \frac{S_n}{S_0}$ $\qquad \underline{A^n_i = \{S_i \neq 0\}}$

$T_j = \frac{S_j}{S_0}$

Hibert Nullstellensatz $\Longrightarrow$ $G_i(T_1, \cdots, T_n)$ s.t. $\sum_{i=1}^r \underbrace{F_i(1, T_1, \cdots, T_n)} \cdot \underbrace{G_i(T_1, \cdots, T_n)} = 1$

$\Rightarrow S_0^{l_0} = \sum \underbrace{F_i(S_0, S_1, \cdots, S_n)}_{\in I} \cdot \underbrace{\left(G_i\left(\frac{S_1}{S_0}, \cdots, \frac{S_n}{S_0}\right)\right) S_0^{m_0}}$

Similarly $\Rightarrow$ $S_i^{l_i} \in I$. $\forall i = 0, \cdots, n$.

$S^m = S_0^{m_0} S_1^{m_1} \cdots S_n^{m_n}$ $\qquad |m| \geq N \Rightarrow \exists\, m_i \geq l_i$

$\Rightarrow \boxed{S^m \in I}$

$S^m = S_i^{l_i} \cdot \left(S_0^{m_0} \cdots S_i^{m_i - l_i} \cdots S_n^{m_n}\right)$

$\Rightarrow I_m \subseteq I$.

Def: A quasi-projective variety is an open subset of a closed projective set.

This includes: closed subset of $\mathbb{A}^n$

- - of $\mathbb{P}^n$

$X \setminus Y$

closed subsets of $\mathbb{P}^n$

subvariety $Y \subset X$: $Y = Z \setminus Z_1$, Z, Z_1 closed subsets of X.

Def: affine variety: quasi-proj. that is isomorphic to a closed subset in $\mathbb{A}^n$

quasi-affine var: open subset of affine variety.

rational functions: $f(s_0, \dots, s_n) = \dfrac{P(s_0, \dots, s_n)}{Q(s_0, \dots, s_n)}$

(P, Q homogeneous, $\deg P = \deg Q$)

$\dfrac{P_1}{Q_1} = \dfrac{P_2}{Q_2} \Leftrightarrow P_1 Q_2 - P_2 Q_1$ vanish on X

f is regular at $x \in X$ if $\exists$ representation $f = \frac{P}{Q}$ s.t. $Q(x) \neq 0$.

$\mathrm{Dom}(f) = \{x : f \text{ is regular at } x\} \subset X$.

f is regular on X if f is regular at every $x \in X$.

$k[X] = \{f : f \text{ is regular fct. on } X\}$

- <u>Fact</u>: if X is an irred. closed projective set. then $k[X] = k$ i.e. only constant fcts.

($\rightsquigarrow$ holomorphic fcts on compact complex mfd. are constants)

<u>Ex</u>: $X = \mathbb{P}^n$. $f = \frac{P}{Q}$ is not regular at x s.t. $Q(x) = 0$.