$$\left\{\text{closed subsets of } \mathbb{A}^n\right\} \longleftrightarrow \left\{ I \text{ ideals of } k[T] \text{ s.t. } I = \sqrt{I} \right\}$$

X

$$\updownarrow$$

$$\left\{ k[X] = k[T]/I_X \text{ with no nonzero nilpotent} \right\}$$

$0 = f^r \in k[X] \Rightarrow f = 0.$

$$X \mapsto I(X) = \{F \in k[T] : F|_X \equiv 0\}$$

$f : X \to Y$ regular map

$$\updownarrow$$

$f^* : k[Y] \to k[X]$ homomorphism of k-algebra.

- $Y \subset X \iff k[X] \to k[Y]$ surjective morphism

$I(Y) \supset I(X)$ $\qquad F|_X \mapsto F|_Y$

$$k[T]/I_X \twoheadrightarrow k[T]/I_Y$$

- Prop: $f : X \to Y$. regular map. (closed subset)

$k[Y] \to k[X]$ is injective iff $\overline{f(X)} = Y$.

Pf: $\underline{\forall u \in k[Y],\ f^*u \in k[X],\ f^*u = 0 \Rightarrow u = 0.}$

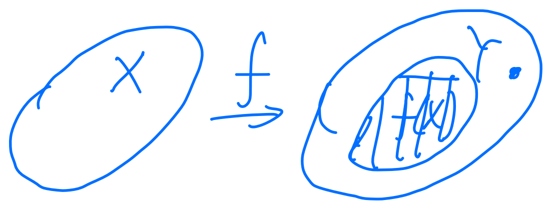
$\forall u \in k[Y],\ f^*u = 0 \iff u$ vanishes on $f(X)$

$\iff u$ vanishes on $\overline{f(X)}$

$\iff \underline{u \in I(\overline{f(X)})}$

Injectivity: $\left(u \in I(\overline{f(X)}) \Leftrightarrow u \in I(Y)\right) \Leftrightarrow I(\overline{f(X)}) = I(Y)$.

$Z(I(\overline{f(X)})) = Z(I(Y)) \Rightarrow \overline{f(X)} = Y$



Jacobian Conj. Map: $\Phi: A^2 \to A^2$

$(x,y) \mapsto (f(x,y), g(x,y))$

polynomials

$\det\left(\frac{\partial(f,g)}{\partial(x,y)}\right) \neq 0$. Then Φ is an automorphism of A^2

$Aut(A^2) = \left\{ \begin{array}{l} \text{linear affine maps. } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v \\ (x' = ax, \ y' = \beta y + f(x)) \end{array} \right\}$

Rational Maps   Rational Function

Irreducible closed set

X Reducible if $X = X_1 \cup X_2$ $X_1, X_2 \subsetneq X$ closed

Ex: $X = \{F = 0\}$, X irr. $\Leftrightarrow$ F is irred.

Thm: Any closed set has a unique decomposition into irredundant union of irreducible closed sets.

Pf: X if reducible $X = X_1 \cup X'$.

X_1 if reducible $X_1 = X_2 \cup X_2'$.

- - - .

$X \supsetneq X_1 \supsetneq X_2 \supsetneq$

$\Updownarrow$

$I(X) \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \cdots$ Ascending chain of ideals

$\xRightarrow{k[T] \text{ Noetherian}}$ Can't have infinite strictly increasing chain. So existence

Uniqueness: $X = \bigcup_i X_i = \bigcup_j Y_j$

$X_i = \bigcup_j (X_i \cap Y_j) \xRightarrow{X_i \text{ irred.}} X_i \cap Y_j = X_i$ for some j

$X_i \subseteq Y_j$

$Y_j \subseteq X_{i'} \Rightarrow X_i \subseteq Y_j \subseteq X_{i'} \xRightarrow{\text{irredundant}} X_i = X_{i'}$

$Y_j = X_i$

$\Rightarrow \{X_i\} \leftrightarrow \{Y_j\}$ same decomposition.

Prop: X is reducible $\iff$ $k[X]$ has zero divisors.

$(0 \neq u,\ v \neq 0 \Rightarrow u \cdot v = 0)$

Pf: • X is reducible $\Rightarrow X = X_1 \cup X_2$, $X_i \subsetneq X$.

$X_i \subsetneq X$: $F_i \in I(X_i) \setminus I(X)$. $\quad F_1 \cdot F_2 = 0$ on X

$I(X_i) \supsetneq I(X)$ $\quad \Rightarrow k[X]$ has zero divisors.

• $k[X]$ has zero divisor. $\quad F_1 \cdot F_2 = 0$ on X, $F_i \neq 0$ on X.

$$X = (\{F_1 = 0\} \cap X) \cup (\{F_2 = 0\} \cap X)$$

$\Rightarrow X$ is reducible.

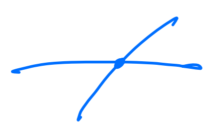
X is irred. $\iff \underset{\displaystyle u, v}{\overset{\displaystyle k[T]/I(X)}{\overset{\shortparallel}{k[X]}}}$ has no zero divisors.

$\quad u,v \in k[X]$ $\left(\begin{matrix} u \cdot v = 0 \\ u \neq 0 \end{matrix} \Rightarrow v = 0\right)$

$\iff I(X)$ is a prime ideal

$(F \cdot G \in I \Rightarrow F \in I$ or $G \in I)$.

Thm: Product of Irred. closed sets is irreducible.



$k[T] \ni$

$\{F_1 = \cdots = F_k = 0\} = X \subseteq A^m$

$\{G_1 = \cdots = G_\ell = 0\} = Y \subseteq A^n$

$\in k[U]$

$$\Rightarrow \underset{\displaystyle \{F_1(T) = \cdots = F_k(T) = 0 = G_1(U) = \cdots = G_\ell(U)\}}{\underset{\shortparallel}{X \times Y}} \subseteq A^{m+n}$$

Def: · X irreducible closed sets.

quotient field = field of fractions

$k(X) = \{\text{field of rational functions}\} = Q(k[X])$

$$= \left\{ \frac{f}{g} : f, g \in k[X] \right\}$$

$$= \left\{ \frac{F_1}{G_1} \sim \frac{F_2}{G_2} : F_1 G_2 - F_2 G_1 \in I(X) \right\} \quad F_i, G_i \in k[T]$$

· $\varphi = \frac{f}{g} \in k(X)$ is regular at $x \in X$ if $g(x) \neq 0$.

Thm: A rational function φ that is regular at all points of X is a regular function.

$x^2 + y^2 = 1 \Leftrightarrow (1+x)(1-x) = y^2$

$$\frac{y}{1-x} = \frac{1+x}{y}$$

$$\left(\frac{p_1}{q_1} = \frac{p_2}{q_2} \Leftrightarrow p_1 q_2 - p_2 q_1 = 0 \right)$$

Pf: $\varphi \in k(X)$, $\forall x \in X$.

$$\varphi = \frac{f_x}{g_x} \quad g_x(x) \neq 0,$$

$$I = (g_x, x \in X) = (g_{x_1}, \cdots, g_{x_N})$$

$\cap\!\!\!\!\;\;$ $k[X]$

Nullstellensatz

$\Rightarrow Z(g_{x_1}, \cdots, g_{x_N}) = Z(I) = \phi \Longrightarrow I = k[X]$

$\Rightarrow \exists\, u_1, \cdots, u_N \in k[X]$ s.t. $\sum_{i=1}^{N} u_i \cdot g_{x_i} = 1$

$\Rightarrow \varphi = \sum_{i=1}^{N} u_i \cdot g_{x_i} \cdot \varphi = \sum_{i=1}^{N} u_i \cdot f_{x_i} \in k[X]$ ∎

$\varphi \in k(X)$, $\quad \varphi = \frac{f}{g}$, $\quad \{x : g(x) \neq 0\}$ open subset of X

$\quad\quad = \frac{f_i}{g_i} \quad \underline{U_i := \{x : g_i(x) \neq 0\}}$

$U_\varphi = \bigcup_i U_i = \{x : \varphi \text{ is regular}\}$: domain of definition of φ
non-empty subset.

$\boxed{\varphi_1, \dots, \varphi_m : \text{ rational functions}}$

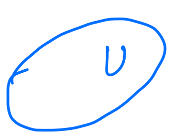
$\{x : \varphi_i \text{ is regular at } x\} = \underline{U_{\varphi_1} \cap \cdots \cap U_{\varphi_m} \overset{\neq \phi}{} \text{ open subsets.}}$

If $U_{\varphi_1} \cap \cdots \cap U_{\varphi_m} = \phi$, then $X = X \setminus (U_{\varphi_1} \cap \cdots \cap U_{\varphi_m})$

$$= \bigcup_{i=1}^{m} \underbrace{(X \setminus U_{\varphi_i})}_{\text{closed}}$$

$\overset{X \text{ irred.}}{\Longrightarrow}$ $X = X \setminus U_{\varphi_i}$ for some i $\Rightarrow$ $U_{\varphi_i} = \phi$ for some i.

$\underline{\text{Cor}}$: A rational function is uniquely determined if it is specified by some non-empty open subset $U \subset X$.



$\varphi_1 = \varphi_2$ on $U \Rightarrow \varphi_1 = \varphi_2$ in $k(X)$.

$\varphi_2 - \varphi_1 = \varphi = 0$ on $U \Rightarrow \varphi = 0$ in $k(X)$.

$\underline{\text{pf}}$: $\underline{\varphi(x) = 0,\ \forall x \in U}$, $\varphi \neq 0$ in X

$\varphi = \frac{f}{g}$, $\quad X_1 = X \setminus U$, $\quad X_2 = (f = 0)$

$(f(p) \neq 0 \Rightarrow \varphi(p) \neq 0 \Rightarrow p \notin U)$ $\quad X = X_1 \cup X_2$ contradicts to irred. of X ∎

Rational Map : $\varphi : X \to \mathbb{A}^m$ is defined on $U \overset{\text{open}}{\subset} X$

$x \mapsto (\varphi_1(x), \cdots, \varphi_m(x))$

$U = \bigcap_{i=1}^{m} \mathrm{Dom}(\varphi_i) = \bigcap_{i=1}^{m} U_{\varphi_i}$

$\underbrace{\varphi : X \to Y}$: $\forall x \in X$ where φ is regular $\varphi(x) \in Y$.

$$\boxed{\varphi(X) = \{\varphi(x) : x \in X, \varphi \text{ is regular at } x\}}$$

- $Y = (\underline{G_1 = \cdots = G_s = 0})$. $I_Y \ni G$

$G(\underbrace{\varphi_1, \cdots, \varphi_m}) = 0$ on X

- $\varphi(X)$ is dense in Y. $\varphi : U \to \varphi(X) \subset Y$

$\varphi^* : k[Y] \hookrightarrow k(X) \rightsquigarrow \boxed{\varphi^* : k(Y) \hookrightarrow k(X)}$

$k(Y) = Q(k[Y])$

$u|_Y \mapsto u(\varphi_1, \cdots, \varphi_m) \in k(X)$