Def: closed subset of $\mathbb{A}^n$: $X \subset \mathbb{A}^n$, $F_1, \cdots, F_m \in \underbrace{k[T_1, \cdots, T_n]}_{= k[T]}$

$X = \{ a \in \mathbb{A}^n : F_1(a) = \cdots = F_m(a) = 0 \}$

$I \subseteq k[T_1, \cdots, T_n]$ ideal. $F, G \in I$, $F + G \in I$
$F \in I, G \in k[T]$, $F \cdot G \in I$

Thm (Hilbert basis thm) I is finitely generated:

$F \in I = (F_1, \cdots, F_m)$, $F = \sum_{i=1}^{n} F_i \cdot G_i$, $G_i \in k[T]$

$$Z(I) = \{ a \in \mathbb{A}^n : F(a) = 0, \forall F \in I \}$$
$$= \{ a \in \mathbb{A}^n : F_1(a) = \cdots = F_m(a) = 0 \}$$

$I \longrightarrow Z(a)$ closed subset.

subset $X \longmapsto \underline{I(X) = \{ F \in k[T] : F|_X \equiv 0 \}}$ ideal.

$$\left(\begin{aligned} & k[T] \to k[X] = k[T]/I(X) \quad \text{coordinate ring.} \\ & \qquad\qquad\quad = \{ f : X \to k \text{ s.t. } f = F|X \} \\ & F \mapsto F|X. \end{aligned} \right)$$

Properties:
- $X \subseteq Y \Longrightarrow I(X) \supseteq I(Y)$
- $I_1 \subseteq I_2 \Longrightarrow Z(I_1) \supseteq Z(I_2)$

Thm: 1. $\underbrace{Z(I(X)) = \overline{X}}$ (closure in the Zariski topology) closed subset

2. $I(Z(I)) = \underset{\sqrt{I}}{\mathrm{rad}(I)} = \{ F \in k[T] : f^r \in I \text{ for some } r \geq 0 \}$. $r \in \mathbb{Z}_{\geq 0}$.

Assume: k is algebraically closed

Pf: 1. $\underset{\underset{p}{\cup\!\!\!\shortmid}}{X} \subseteq Z(\underbrace{I(X)})$: $\overset{\forall p \in X}{\forall F \in I(X)}$, $F(p)=0 \Rightarrow p \in Z(I(X))$

$\Rightarrow \bar{X} \subseteq Z(I(X))$

$q \notin \bar{X} = \{\alpha \in \mathbb{A}^n : F_1(\alpha) = \cdots = F_m(\alpha) = 0\}$, $\exists F_i$, $F_i(q) \neq 0$.

$(F_1, \cdots, F_m) = I(X) \quad \Rightarrow q \notin Z(I(X))$.

$\bar{X}^c \subseteq Z(I(X))^c \Rightarrow \bar{X} \supseteq Z(I(X))$

2. $I = (F_1, \cdots, F_m)$. $G \in k[T]$

$G(\alpha) = 0$ whenever $F_1(\alpha) = \cdots = F_m(\alpha) = 0$

Then $G^r \in I$ for some $r \in \mathbb{Z}_{>0}$.

$\quad \| $

$\sum_i F_i \cdot H_i \quad H_i \in k[T]$.

Hilbert's Strong $\underbrace{\text{Nullstellensatz}}_{\text{Theorem of zero}}$

$\boxed{\text{Weak Nullstellensatz: (Version I)}}$ $I \subset k[T]$, $I \neq k[T]$

$k[T] = k[T_1, \cdots, T_n]$

Then $Z(I) \neq \phi$.

Ex: $k[T] = \underbrace{k[T_1]}$, $I = (f) \neq k[T_1] \Leftrightarrow f$ not a non-zero constant

$\Rightarrow Z(f) \neq \phi$. (since k is algebraically closed)

Version II. If $\underset{k[T]}{m}$ is a maximal ideal of $k[T]$, $\boxed{m = (T_1 - \alpha_1, \cdots, T_n - \alpha_n)}$ for some $(\alpha_1, \cdots, \alpha_n) \in k^n$.

Fact: $I \subseteq m$ maximal ideal, I in $k[T]$ $Z(I) \supseteq Z(m) = \{(a_1, \cdots, a_n)\}$

Proof of Strong Nullstellensatz

$I = (F_1, \cdots, F_m)$. $G(a) = 0 \ \forall a \in Z(I) \Rightarrow G^r \in I$.

$G \neq 0$. Introduce a new variable U.

$G \in k[T]$ $\boxed{F_1, \cdots, F_m, UG-1} \in k[T_1, \cdots, T_n, U]$, $F_1, \cdots, F_m \in k[T_1, \cdots, T_n]$

$T_i \mapsto T_i$

$U \mapsto \frac{1}{G(T)}$

$\Rightarrow P_1, \cdots, P_m, Q \in k[T_1, \cdots, T_n, U] \to k(T_1, \cdots, T_n)$, where $T = (T_1, \cdots, T_n)$

$1 = P_1 F_1 + \cdots + P_m F_m + Q(UG-1)$. [weak Nullstellensatz].

$1 = P_1(T,U) F_1(T) + \cdots + P_m(T,U) F_m(T) + Q(T,U) \cdot (U \cdot G(T) - 1)$

$U \rightsquigarrow \frac{1}{G(T)}$

$1 = \left(P_1\left(T, \frac{1}{G(T)}\right) \cdot F_1(T) + \cdots + P_m\left(T, \frac{1}{G(T)}\right) \cdot F_m(T)\right) + Q\left(T, \frac{1}{G(T)}\right)\left(\frac{1}{G(T)} G(T) - 1\right)$

$$1 = \frac{S_1(T)}{G(T)^{d_1}} F_1(T) + \cdots + \frac{S_m(T)}{G(T)^{d_m}} \cdot F_m(T)$$

$\Rightarrow G(T)^N = R_1(T) F_1(T) + \cdots + R_m(T) \cdot F_m(T) \in \sqrt{I}$.

$I = m$ maximal ideal $\quad I(Z(I)) = m$

$\overset{m}{\shortparallel}$

$Z(I) \ni p = (a_1, \cdots, a_n)$

$I(Z(I)) \subseteq I(P) = \underline{(T_1 - a_1, \cdots, T_n - a_n)}$.

Weak Null. $\quad I \subsetneq \underset{}{\overset{R}{\overset{\shortparallel}{k[T]}}} \Rightarrow Z(I) \neq \phi$

$I \subseteq m \subsetneq k[T] \quad$ Need show $Z(m) \neq \phi$

$Z(I) \supseteq Z(m)$

$E = R/m$ is $\underline{\text{a field}}$ that finitely generated k-algebra.

$k[T] \to k[T]/m = k[\underbrace{x_1, \cdots, x_n}]$

$T_i \mapsto x_i$

$\boxed{\text{Key. } E \text{ is a finite algebraic extension of } k. \left(\begin{array}{l} \text{i.e. } E \text{ is finitely} \\ \text{generated as a} \\ k\text{-module} \end{array} \right) \quad (\text{Artin-Tate}) \quad [E : k] = \dim E/k < +\infty .}$

$k(t)$ transcendental extension of k

$\{1, t^k\}_{k \in \mathbb{Z}}$

k - algebraically closed $\Rightarrow E = k$.

$k[T] \longrightarrow k[T]/m \cong k$

$T_i \longmapsto a_i \in k$.

$m \subseteq (T_1 - a_1, \cdots, T_n - a_n) \subset k[T]$

$\Rightarrow \quad Z(m) = (a_1, \cdots, a_n) \subseteq Z(I)$

$\neq \phi$

∎

$$\{\underbrace{\text{closed subsets of } \mathbb{A}^n}\} \longleftrightarrow \{\underbrace{\text{Ideal } I \subseteq k[T],\ I = \sqrt{I}}\}$$

$$\Updownarrow$$

$$\{ k[X] = k[T]/\sqrt{I} \}$$

has no nilpotents

$$\left(0 = f^r \in k[X] \Rightarrow f^r \in \sqrt{I} \Rightarrow f \in \sqrt{I} \Rightarrow f = 0\right)$$

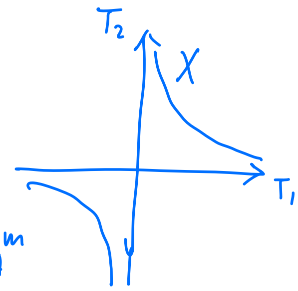
$$\overline{X} \subseteq \overline{Y} \iff I(X) \supseteq I(Y)$$

Ex: $X = \{(a_1, a_2) : a_1 \cdot a_2 = 1\}$ $\quad F(T_1, T_2) = T_1 \cdot T_2$

$X \subset \mathbb{A}^2$, $\quad a_1 \cdot a_2 = F(a_1, a_2)$

$$k[X] = k[T_1, T_2]/(T_1 \cdot T_2) = k[T_1, T_1^{-1}] = \left\{ \frac{G(T_1)}{T_1^n},\ G(T_1) \in k[T_1] \right\} \quad n \geq 0.$$

$$I(X) = (T_1 \cdot T_2)$$



- <u>Regular Maps</u>: $\quad \mathbb{A}^n \supseteq X \xrightarrow{f} Y \subseteq \mathbb{A}^m$, $\quad x \mapsto y$ $\quad$ <u>a map</u>.

$\exists$ m <u>regular function $f_1, \ldots, f_m$</u> s.t. $\in k[X] = k[T]/I(X)$

$$f(x) = (f_1(x), \ldots, f_m(x)).$$

$Y = Z((G_1, \ldots, G_k))$. $\quad$ <u>$G_i(f_1, \ldots, f_m) = 0$</u> $\in k[X]$, $\forall\ i = 1, \ldots, k$.

Ex: $X = \{(a_1,a_2) : \underline{a_1 a_2 = 1}\} \xrightarrow{f} Y = \mathbb{A}^1$

$(a_1,a_2) \longmapsto a_1$

$T_1(a_1,a_2)$

$\operatorname{Im}(f) = \mathbb{A}^1 \setminus \{0\}$.

- $X \xrightarrow{f = (f_1,\dots,f_m)} Y$.

$$\begin{array}{ccc}
X & & Y \\
\wr & & \wr \\
k[X] & \xleftarrow{f^*} & k[Y] \\
\| & & \| \\
k[S_1,\dots,S_n]/I(X) & & k[T_1,\dots,T_m]/I(Y). \\
 & & \cup \\
f^* u = u(f_1(s),\dots,f_m(s)) & \longleftarrow\!\shortmid & u(T_1,\dots,T_m)
\end{array}$$

$[T_i] = t_i \mapsto f_i(x)$

$\varphi : k[Y] \to k[X]$ a $\boxed{\text{homomorphism of } k\text{-algebra}}$

$$\begin{array}{ccc}
\cup & & \cup \\
k & \xrightarrow{Id} & k
\end{array}$$

is of the form $\varphi = f^*$ for some regular map.

$f(x) = (f_1(x),\dots,f_m(x))$.

$\left\{\dfrac{\text{closed subsets of affine sp.}}{\text{regular maps } f: X \to Y}\right\} \longleftrightarrow \left\{\dfrac{\text{f.g. } k\text{-alg. with no nilpotents}}{\text{homomorphism of } k \text{ algebra}}\right\}$

$X \to Y \quad \longleftrightarrow \quad k[X] \leftarrow k[Y]$

$\{$ Topological spaces $\} \longleftrightarrow \{$ algebra of functions $\}$

$X \longleftrightarrow \underline{C(X, \mathbb{R})}$

$R \rightsquigarrow R_\varepsilon$