Projective Plane:

$$\mathbb{P}^2_k = \{[\xi,\eta,\zeta] : (\xi,\eta,\zeta)\neq(0,0,0)\} \supset \mathbb{A}^2_k$$

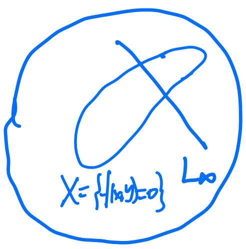
$$[\xi,\eta,\zeta] \overset{\lambda\neq 0}{=} [\lambda\xi,\lambda\eta,\lambda\zeta]$$

$$\| $$

$$\mathbb{A}^2_k \cup L_\infty \qquad [x,y,1] \longleftarrow\!\shortmid (x,y)$$

$$[\xi,\eta,\zeta] \mapsto \left(\frac{\xi}{\zeta},\frac{\eta}{\zeta}\right)$$

$$L_\infty = \{\zeta=0\} = \{[\xi,\eta,0] : (\xi,\eta)\neq(0,0)\}$$



$$f(x,y)=0 \qquad \deg f = n$$

$$\| $$

$$f\left(\frac{\xi}{\zeta},\frac{\eta}{\zeta}\right) \qquad \zeta^n\cdot f\left(\frac{\xi}{\zeta},\frac{\eta}{\zeta}\right) = F(\xi,\eta,\zeta)$$

homogeneous poly. of deg n.

$$X \subset \overline{X} = \{[\xi,\eta,\zeta] : F(\xi,\eta,\zeta)=0\}$$

$$F(\lambda\xi,\lambda\eta,\lambda\zeta) = \lambda^n F(\xi,\eta,\zeta).$$

$$F(\xi,\eta,\zeta) \rightsquigarrow f(x,y) = F(x,y,1).$$

$$\overline{X} = \{F(\xi,\eta,\zeta)=0\} \qquad \overline{X}\cap \mathbb{A}^2_k = X = \{f(x,y)=0\}$$

$$X = \{(x-a)^2+(y-b)^2 = r^2\}$$

$$\zeta^2\left(\left(\frac{\xi}{\zeta}-a\right)^2+\left(\frac{\eta}{\zeta}-b\right)^2 - r^2\right) = \underline{(\xi-a\zeta)^2+(\eta-b\zeta)^2-r^2\zeta^2} = 0.$$

$$\| $$

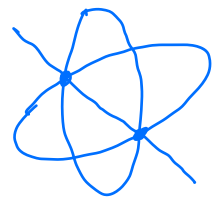
$$F(\xi,\eta,\zeta)$$

$$\overline{X}\cap L_\infty = \{F(\xi,\eta,\zeta)=0,\ \zeta=0\}$$

$$= \{\xi^2+\eta^2=0,\ \zeta=0\} = \{[1,i,0],\ [1,-i,0]\}.$$

Thm (Bezout) $X, Y \subset \mathbb{P}^2$. algebraic. ($\underline{X \not\subset Y} \Leftrightarrow f \nmid g$)

$\{F(\xi,\eta,\zeta)=0\}$ $\{G(\xi,\eta,\zeta)=0\}$



Assume $\underline{X \text{ is smooth}}$. (non-singular).

$$\sum_{P \in X \cap Y} i(P; X, Y) = \underbrace{\#(X \cap Y)}_{\text{counting multiplicity}} = \underset{\deg F}{\deg X} \cdot \underset{\deg G}{\deg Y}.$$

- $P \in X \rightsquigarrow X \cap \mathbb{A}^2_k = \{F(x,y,1)=0\}$, $F(x,y,1) = f(x,y)$

$P = [\xi, \eta, \zeta]$, $\zeta \neq 0$

$(\alpha,\beta) = P$ nonsingular. if $f'_x(\alpha,\beta) \neq 0$ or $f'_y(\alpha,\beta) \neq 0$.

$f'_y(\alpha,\beta) \neq 0 \Rightarrow t = x-\alpha$ local parameter, t regular at P.
(Coordinate)

$\Downarrow$ IFT

$y = y(x)$ any rational function $u = \frac{p(x,y)}{q(x,y)}$. $q(P) \neq 0$

$u = t^k \cdot v$. $v(P) \neq 0$.

$$\boxed{k = \mathrm{ord}_P(u) \in \mathbb{Z}}$$

$Y = \{G(\xi,\eta,\zeta)=0\}$

- $i(P; X, Y) = \mathrm{ord}_P(g) > 0$, $P \in X \cap Y$.

$g(x,y) = G(x,y,1)$

$g|_X = t^{k} \cdot v$

- Special Case: X is a line L

$X \cap Y = L \cap Y$ is finite

(Lem: f irred. $f \nmid g$, then $\{ f(x,y)=0=g(x,y) \}$ finite solution)

- $\underline{L_\infty = \{\zeta=0\} \text{ does not pass through pts. of intersection.}}$
- $X = \{\eta=0\}$
- $Y \cap \mathbb{A}^2 = \{ \overset{G(x,y,1)}{\overset{\shortparallel}{g(x,y)}} = 0 \}$, $X \cap \mathbb{A}^2 = \{ y=0 \}$

$$\phantom{Y \cap \mathbb{A}^2 = \{} \overset{\shortparallel}{\underbrace{g_0 + g_1(x,y) + \cdots + g_n(x,y)}} \qquad g_n(x,y) = \sum_{k=0}^{n} a_k x^k \cdot y^{n-k}$$

x is local parameter for $X \cap \mathbb{A}^2$

$$X \cap Y = \{ (x,0) : \underline{g_0 + g_1(x,0) + \cdots + g_n(x,0) = 0} \}$$

$$\#(X \cap Y) = \deg(g_0 + g_1(x,0) + \cdots + g_n(x,0)) = n .$$

$$g_n(x,0) = a_n \cdot x^n \neq 0 .$$

- $G(\xi,\eta,\zeta) = g_0 \cdot \zeta^n + g_1(\xi,\eta)\zeta^{n-1} + \cdots + g_{n-1}(\xi,\eta)\zeta + g_n(\xi,\eta)$

$$\underline{G(1,0,0)} = g_n(1,0) = a_n \neq 0$$

$\cancel{0}$ ⇐ $(1,0,0) \notin X \cap Y$.

$$X \cap Y = (\eta=0) \cap (G(\xi,\eta,\zeta)=0).$$

∎

· $k = \mathbb{C}$, X nonsingular. $Y \not\supset X$.

$$Y = \{G(\xi,\eta,\zeta)=0\}$$

$$\deg Y = \deg G = n.$$

$$Y \rightsquigarrow n \cdot \text{lines} = L_1 + \cdots + L_n = \{H(\xi,\eta,\zeta)=0\}$$

$$H = H_1 \cdots H_n$$

$$\#(X\cap Y) = \#(L_1\cap X) + \cdots + \#(L_n \cap X)$$

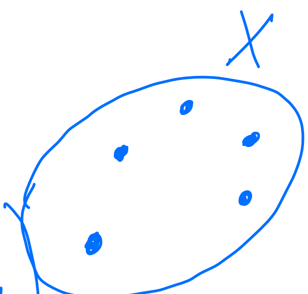
$$= \deg X + \cdots + \deg X = \boxed{\deg X \cdot \deg Y}$$

$$u = \frac{H}{G}$$ rational fct. on $\mathbb{P}^2$
on X

$\forall P \in X\cap Y$

$u = t^k \cdot v$, v regular $v(P)\neq 0$

$$\mathrm{Div}(u) = \sum_{P\in X} \mathrm{ord}_P(u)\cdot P$$ (divisor associated to rational fct.)

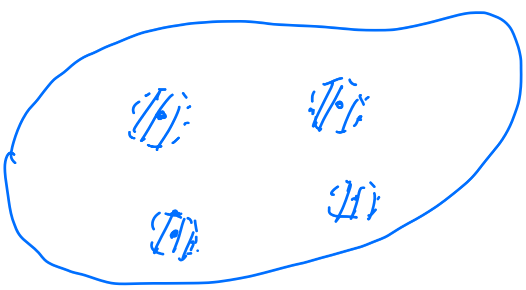


$$\boxed{\deg(\mathrm{Div}(u)) = \sum_{P\in X} \mathrm{ord}_P(u) = \#(X\cap Y') - \#(X\cap Y) \overset{?}{=} 0}$$

<u>Fact:</u> # zeros = # poles for any rational fct. on X.

$$\frac{1}{2\pi i}\int_C \frac{du}{u} = \#\text{zeros} - \#\text{poles}.$$

$$u = t^k\cdot v \Rightarrow \frac{du}{u} = k\cdot\frac{dt}{t} + \frac{dv}{v}$$



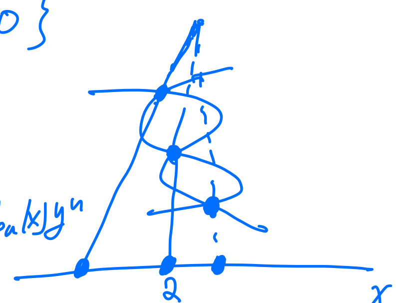
$$d\left(\frac{du}{u}\right) = d\left(\left(\frac{\partial u}{\partial t}\cdot\frac{1}{u}\right)dt\right) = 0$$

$$\int_{\cup C_i}\frac{du}{u} = \int_{X\setminus \cup D_i} d\left(\frac{du}{u}\right) = 0.$$

$\blacksquare$

- Sketch of alg. proof using <u>resultant</u>

$$X = \{\underset{\substack{\cap \\ k[x][y]}}{f(x,y)} = 0\}, \quad Y = \{\underset{\substack{\cap \\ k[x][y]}}{g(x,y)} = 0\}$$



$$f(x,y) = a_0(x) + a_1(x)\cdot y + \cdots + a_m(x)y^m, \quad g(x,y) = b_0(x) + b_1(x)\cdot y + \cdots + b_n(x)y^n$$

$$\mathrm{Result}_y(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_m & & & \\ & a_0 & \cdots & a_{m-1} & a_m & & \\ & & \ddots & & & \ddots & \\ & & & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & \\ & & \ddots & & & \ddots & \\ & & & b_0 & \cdots & \cdots & b_n \end{vmatrix}$$

$(m+n)\times(m+n)$

$\mathrm{Result}_y(f,g)(a) = 0 \iff f(a,y) = g(a,y) = 0$ has a common solution

$$\deg\left(\mathrm{Result}(f,g)(x)\right) = m \cdot n = \deg X \cdot \deg Y. \qquad \blacksquare$$

Closed subsets of $\mathbb{A}^n_k$.

$$X = \{ F_1(T) = \cdots = F_n(T) = 0 \}$$

equations of X.

$$= \{ \underset{\alpha}{(\alpha_1, \cdots, \alpha_n)} : G(\alpha) = 0, \ G \in \underbrace{(F_1, \cdots, F_n)}_{F_1 \cdot G_1 + \cdots + F_n G_n} \}$$

$$\underline{(\{F_\tau\}_{\tau \in A})} \ \underset{\text{Basis Thm}}{\overset{\text{Hilbert}}{=}} \ \underline{(F_1, \cdots, F_n)}$$

Open subset : $U = \mathbb{A}^n_k \setminus X$.

- Intersections of closed subsets are closed
- Finite unions of . . . . closed.

$$X_1 = (F_1 = \cdots = F_m = 0)$$
$$X_2 = (G_1 = \cdots = G_n = 0)$$

$\rightsquigarrow$ Zariski topology.

$$X_1 \cup X_2 = \left(F_i \cdot G_j = 0, \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \right)$$

Ex: • closed subsets of $\mathbb{A}^1$: $\mathbb{A}^1$, finite sets.

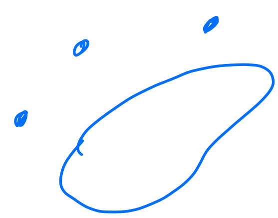
$\{f_1 = \cdots = f_n = 0\}$ empty set.

• closed subsets of $\mathbb{A}^2$

$$\{f_1(x,y) = f_2(x,y) = \cdots = f_k(x,y) = 0\}$$

$$\big(f_i(x,y) = D(x,y) \cdot g_i(x,y)\big)$$

$$= \underbrace{\{g_1(x,y) = \cdots = g_k(x,y) = 0\}}_{\text{finite set}} \cup \underbrace{\{D(x,y) = 0\}}_{\text{affine alg. curve}}$$



• X closed subset of $\mathbb{A}^n$.

$$k[X] = \{f = F|_X : F \in k[x_1, \dots, x_n]\}.$$

coordinate ring of X.

$$I_X \hookrightarrow k[x_1,\cdots,x_n] \longrightarrow k[X] \cong k[x_1,\cdots,x_n]/I_X.$$

$$\| $$

$$\{F|_X \equiv 0 : F \in k[x_1,\cdots,x_n]\}$$

- I ideal of $k[x_1,\cdots,x_n]$

$$\rightsquigarrow Z(I) = \left\{ (a_1,\cdots,a_n) \in \mathbb{A}^n : \begin{array}{c} F(a_1,\cdots,a_n)=0 \\ F \in I \end{array} \right\}$$

$$Z(I) \longleftarrow\!\shortmid\; I$$

$$\{\text{closed subsets}\} \rightleftarrows \{\text{ideal of } k[x_1,\cdots,x_n]\}$$

$$X \longmapsto I_X$$

Thm:
- $Z(I(X)) = X$.
- $I(Z(I)) = \sqrt{I} = \{F \in k[x_1,\cdots,x_n], F^r \in I\}$.

Ex: $I = (x^r)$, $Z(I) = (x=0) \subset \mathbb{A}^1$

$$I(Z(I)) = I(x=0) = (x) = \sqrt{I}$$

Thm (Nullstellensatz) $G, F_1, \cdots, F_m \in k[T_1, \cdots, T_n]$

G is 0 at all solutions of the system

$$F_1 = \cdots = F_m = 0.$$

then $G^N \in (F_1, \cdots, F_m)$ for some $N \geq 0$.

$$\{\text{closed subsets of } \mathbb{A}^n\} \longleftrightarrow \left\{\begin{array}{c}\text{ideals } I \text{ of } k[T_1, \cdots, T_n] \\ I = \sqrt{I}\end{array}\right\}$$