$X = \{ f(x,y) = 0 \} \subset k^2 = \mathbb{A}^2$ affine plane curve. irreducible ($\Leftrightarrow$ f irred.)

<u>Uni-rational</u> : $x = \varphi(t),\ y = \psi(t)$ rational fcts. s.t. $f(\varphi(t), \psi(t)) \equiv 0$ as a fcl. of t.

$\Longleftrightarrow$ $k(X) = \left\{ \dfrac{p(x,y)}{q(x,y)} : f \nmid q \right\} / \sim$ functional field of X

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \Leftrightarrow f \mid p_1 q_2 - q_1 p_2$$

is isomorphic to a subfield of $k(t) = \left\{ \dfrac{p(t)}{q(t)} : p, q \in k[t] \right\}$. $q(t) \not\equiv 0$.

$k(t) = k(\mathbb{A}^1)$.

$\dim X = 1$

<u>Thm</u>: (Lüroth) <u>Unirational</u> $\Longleftrightarrow$ <u>rational</u>

Unirational: $k(X) \hookrightarrow k(\mathbb{A}^1) = k(t)$

rational: $k(X) \cong k(\mathbb{A}^1) = k(t)$

$X = \{ f(x,y) = 0 \} \subset \mathbb{A}^2$, $P \in X$

<u>Def</u>: P is singular : $f_x'(P) = f_y'(P) = f(P) = 0$.

otherwise called nonsingular.

$P = (0,0)$ $\quad f(x,y) = \underbrace{f_r(x,y)}_{\neq 0} + f_{r+1}(x,y) + \cdots + f_n(x,y)$

$\Rightarrow r =$ multiplicity of $P \in X$. $\quad f_k(x,y)$: homogeneous polynomial of $\deg = k$.

. f irred. $\Rightarrow$ $\mathrm{mult}_P X \leq n$.

$\mathrm{mult}_P X = n$. $f_n(x,y) = \prod_{i=1}^{n} (a_i x + b_i y)$ $\quad (k = \bar{k})$

f irred. $p=(0,0)$

$\underline{\text{mult}_p X = n-1.}$ $\quad f(x,y) = f_{n-1}(x,y) + f_n(x,y) \Rightarrow X$ is rational

$\underline{\text{Proof}}$: $\quad y = t\cdot x \qquad \underline{f_{n-1}(x,tx)} + f_n(x,tx) = 0 \quad$ on X

$$\| $$

$$x^{n-1} f_{n-1}(1,t) + x^n f_n(1,t) = 0$$

$$\Rightarrow \quad x = -\frac{f_{n-1}(1,t)}{f_n(1,t)} = \varphi(t), \qquad y = t\cdot x = -\frac{t\cdot f_{n-1}(1,t)}{f_n(1,t)} = \psi(t).$$

$\Rightarrow X$ is rational.

$\Updownarrow$ (Lüroth)

unirational

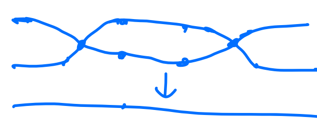
$\text{mult}_p(X) = n-2$: $\quad f(x,y) = f_{n-2}(x,y) + f_{n-1}(x,y) + f_n(x,y)$

$y = t\cdot x \qquad 0 = x^{n-2}\cdot f_{n-2}(1,t) + x^{n-1} f_{n-1}(1,t) + x^n\cdot f_n(1,t)$

$$0 = x^{n-2}\cdot c(t) + x^{n-1}\cdot b(t) + x^n \cdot a(t)$$

$0 = c(t) + b(t)\cdot x + a x^2 \qquad$ birational to

$s = 2a(t)+b \rightsquigarrow \quad \underline{s^2 = p(t)} \,(= b^2 - 4ac) \qquad$ hyperelliptic.



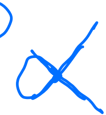
A'_t

• $\text{mult}_p X = 2$

$f(x,y) = f_2(x,y) + f_3(x,y) + \dots$

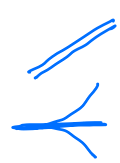
$\| $

$ax^2 + bxy + cy^2$ — $(\lambda_1 x + \mu_1 y)(\lambda_2 x + \mu_2 y)$; $\quad y^2 = x^2 + x^3$



$\underline{\text{node}}$

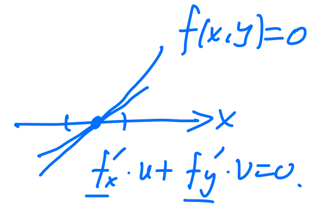
$ax^2 + bxy + cy^2$ — $(\lambda x + \mu y)^2$; $\quad y^2 = x^3$



$\underline{\text{cusp}}$

• Assume P is a nonsingular point.

$$f'_x(P)\neq 0 \quad \text{or} \quad f'_y(P)\neq 0$$



If $\underline{k=\mathbb{R},\ \mathbb{C}}$, then $\underline{\text{implicit fct. thm}}$, $y=y(x)$.

$\{Q(x,y)|_X : Q \text{ polynomial}\}$

Thm: $\forall P\in X$ $\left(= \{f(x,y)=0\}\subset \mathbb{A}^2\right)$ nonsingular, $\exists$ regular fct. t that vanishes at P.

s.t. $\forall$ every $\underline{\text{rational}}$ fct. u that is $\underline{\text{not identically } 0 \text{ on the curve}}$

$$\boxed{u=t^k\cdot v}, \quad k\in\mathbb{Z} \quad \Rightarrow \quad \boxed{k=\mathrm{ord}_P(u)}$$

$\left\{\frac{P}{Q} : f\nmid Q\right\}$ P, Q polynomial

with v is regular at P and $v(P)\neq 0$.

$\left(\mathrm{ord}_P : k(X)\to\mathbb{Z} \text{ valuation}\right)$

(t is $\underline{\text{local parameter} \rightsquigarrow \text{coordinate}}$).

Pf:
$$f(x,y) = \underset{f'_x(0,0)}{\alpha} x + \underset{f'_y(0,0)}{\beta} y + \underbrace{g(x,y)}_{\deg\geq 2}. \qquad \boxed{\beta\neq 0}$$

$P=(0,0)$

$$= \underbrace{x\cdot\varphi(x)}_{\varphi(x)=(\alpha+O(x))} + y\cdot\beta + y\,\underset{\deg\geq 1}{h} \qquad h(0,0)=0$$

On X $\left(=\{f(x,y)=0\}\right)$: $0 = x\cdot\varphi(x) + y\cdot(\beta+h)$ $\Rightarrow$ $y = -\dfrac{x\cdot\varphi(x)}{\beta+h} = x^1\cdot v$.

• $u = \dfrac{p(x,y)}{q(x,y)}$ $\underline{\text{regular}}$ $\quad \underline{p(P)=0}$, $q(P)\neq 0$.

$$p(x,y) = p(x,xv) = x\cdot r(x,v) \Rightarrow u = \frac{x\cdot r}{q} = x\cdot u_1, \qquad u_1=\frac{r}{q} = x\cdot u_2\cdots$$

$\Rightarrow$ $u = x^k\, u_k$ $\quad u_k(0,0)\neq 0$. $\quad$ (v_1, v_2 regular at P)

• $u = \dfrac{u_1}{u_2}$, $\quad u_1, u_2$ regular $\qquad \dfrac{u_1}{u_2} = \dfrac{x^{k_1}\cdot v_1}{x^{k_2}\cdot v_2} = x^{k_1-k_2}\dfrac{v_1}{v_2}$ ∎

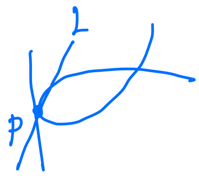
- Intersection number : $X = \{f=0\}$, $Y = \{g=0\}$ — $g(x,y)$; irreducible

$P \in X \cap Y$, P is nonsingular on X $\rightsquigarrow$ local parameter t.

$g = t^k \cdot v \Rightarrow k = \mathrm{ord}_P(g) =$ intersection multiplicity of X and Y at P.
$= i(P; X, Y)$.

$P = (\alpha, \beta)$

$X = \{ \underline{f(x,y) = a(x-\alpha) + b(y-\beta) + g = 0} \}$



L : $\underline{x = \alpha + \lambda t, \; y = \beta + \mu t}$

$i(P; L, X) = \mathrm{mult}_0 \left(a\cdot\lambda t + b\cdot\mu t + g(\alpha+\lambda t, \beta+\mu t) \right)$

> 1 iff $\boxed{a\lambda + b\mu = 0}$

P is nonsingular : $\exists$ a unique such line $\underline{\text{tangent line}}$.

- <u>Projective Plane</u> $\mathbb{P}^2 = (k^3 - \{0\})/\sim$

$(\xi_1, \eta_1, \zeta_1) \sim \lambda(\xi_1, \eta_1, \zeta_1)$

Ex: $\mathbb{C}$
$k = \bar{k}$

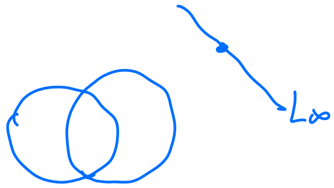
<u>Bézout Thm</u> : X, Y projective curves, X nonsingular not contained in Y. Then

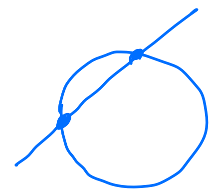
$X = \{F(\xi,\eta,\zeta)=0\}$, $Y = \{G(\xi,\eta,\zeta)=0\}$

$$\boxed{\sum_{P \in X \cap Y} i(P; X, Y) = \deg X \cdot \deg Y = \deg F \cdot \deg G.}$$

Ex: $\{(x,y) : \underbrace{x^2+y^2=1}_{}^{X},\ (x-a)^2+(y-b)^2=r^2\}$

$= \{(x,y) : x^2+y^2=1,\ 2ax+2by = a^2+b^2-r^2\}$





$\subset \mathbb{A}^2 \subset \mathbb{P}^2$

$\mathbb{P}^2 = \mathbb{A}^2 \cup L_\infty$

$\mathbb{A}^2_3 \hookrightarrow \mathbb{P}^2 = \mathbb{A}^2_1 \cup \mathbb{A}^2_2 \cup \mathbb{A}^2_3$

$(x,y) \longmapsto [x:y:1]$ $\quad \underset{\{\xi\neq 0\}}{\|} \quad \underset{\{\eta\neq 0\}}{\|} \quad \underset{\{\zeta\neq 0\}}{\|}$

$$\boxed{\left(\frac{\xi}{\zeta}, \frac{\eta}{\zeta}\right) \overset{\zeta\neq 0}{\longleftarrow\!\shortmid} [\xi,\eta,\zeta]}$$

$$\boxed{(x-a)^2+(y-b)^2=1}$$

$$\zeta^2\left[\left(\frac{\xi}{\zeta}-a\right)^2+\left(\frac{\eta}{\zeta}-b\right)^2=1\right] \Longrightarrow \boxed{(\xi-a\zeta)^2+(\eta-b\zeta)^2-\zeta^2=0}$$

$L_\infty = \{\zeta=0\}$ $\qquad \{\zeta=0,\ \xi^2+\eta^2=0\} = L_\infty \cap X$

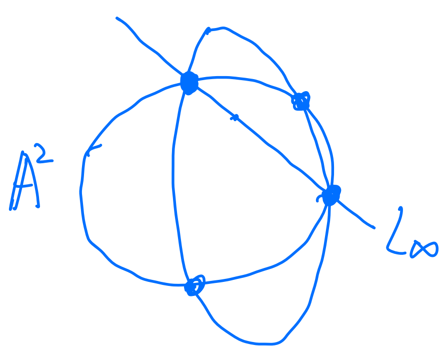
$\|$

$[1,i,0]$

$[1,-i,0]$

$\uparrow$

$\zeta=0$

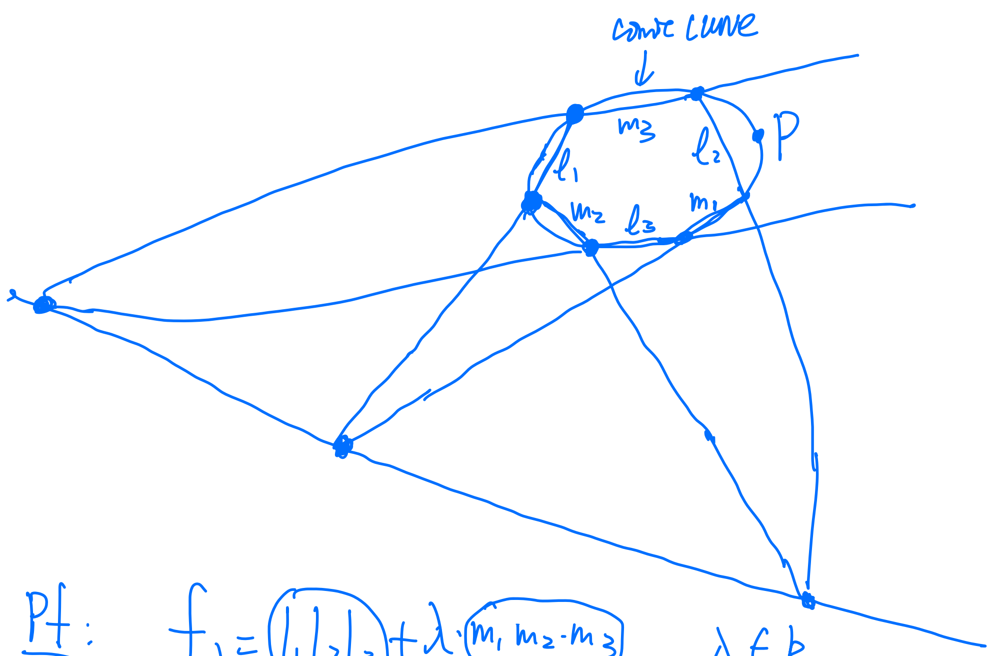


Thm: $X, Y \subseteq \mathbb{P}^N$ $\quad \dim X + \dim Y = N$

X nonsingular $\quad X \not\subseteq Y$.

$$\Rightarrow \quad \sum_{P \in X \cap Y} i(P; X, Y) = \deg X \cdot \deg Y$$

Pascal Thm:



Pf: $f_\lambda = l_1 l_2 l_3 + \lambda \cdot m_1 m_2 \cdot m_3 \quad \lambda \in k$

cubic polynomial

$f_\lambda(P) = 0$.

$Y = (f_\lambda = 0)$, $\quad X =$ conic curve. $/\!/ \; \{g = 0\}$ $\qquad g \nmid f_\lambda$

$\#(X \cap Y) = 6$ counting multiplicity. if $X \subset Y$

$\#(X \cap Y) = 7$

$\Rightarrow g \mid f_\lambda$. $\quad f_\lambda = g \, h$ $\quad \{h = 0\}$ contains $l_1 \cap m_1$, $l_2 \cap m_2$, $l_3 \cap m_3$. ■

linear equation