Algebraic Plane Curves

k field

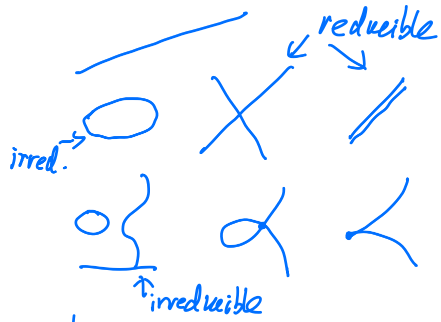
$C = \{(x,y) : f(x,y) = 0\}$ f non constant polynomial

$\deg C = \deg f$.

"=1": line

"=2": conic

"=3": cubic



- $k[x,y]$ is UFD : $f = f_1^{k_1} \cdots f_r^{k_r}$

- <u>Lemma:</u> k field. $f \in k[x,y]$ irred. polynomial, $g \in k[x,y]$

  Assume $f \nmid g$, then the system of eq. $f(x,y) = g(x,y) = 0$ has only a finitely number of solutions.

<u>Cor:</u> For an algebraically closed field k, an irreducible polynomial is uniquely determined by the curve $\underline{f(x,y)=0}$.

$C = \{\underline{f(x,y) = 0 = g(x,y)}\}$ has finite number of solutions by the lemma if $f \nmid g$.

$= \{\underline{f(x,y)=0}\} \leftarrow$ infinite set.

k algebraically closed

<u>Proof of Lemma:</u> $f, g \in k[x,y] \subset k(y)[x]$ $\quad f \nmid g \Rightarrow \boxed{\tilde{u}, \tilde{v} \in k(y)[x]}$

$\underline{a(y)}(f \tilde{u} + g \tilde{v}) = a(y)$.

$\tilde{u}$: $\dfrac{p(y)}{q(y)}$

$f \cdot u + g \cdot v = a(y)$.

$f(\alpha,\beta)=g(\alpha,\beta)=0 \Rightarrow a(\beta)=0 . \Rightarrow$ finitely many β.

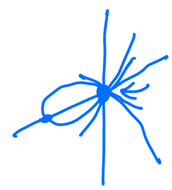
$f(\alpha,\beta)$ has only finitely many sol. in α. ∎

- Rational Curves: An irreducible curve $X=\{f(x,y)=0\}$: . $\exists$ two rational functions $\varphi(t)$, $\psi(t)$ at least one non-constant

$$\varphi(t)=\frac{p_1(t)}{q_1(t)}, \quad \psi(t)=\frac{p_2(t)}{q_2(t)}$$

s.t $\boxed{f(\varphi(t),\psi(t))\equiv 0}$ as an identity in t.

Ex: $y^2=x^2+x^3$.



$y^2=x^2+O(x^2)$

$0=y^2-x^2=(y-x)(y+x)$.

$y=t\cdot x$, $t^2x^2=x^2+x^3$

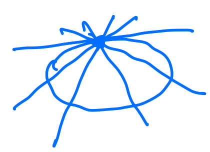
$t^2=1+x \Rightarrow x=t^2-1$, $y=t\cdot(t^2-1)$

$x=t^2-1=\varphi(t)$, $y=t(t^2-1)=\psi(t)$.

Ex: $ay+bx=1$. $x=t$, $y=\frac{1-bt}{a}$ $(a\neq 0)$

Conic: $f(x,y)=0$.

$f(x,y)=ax^2+2bxy+cy^2+dx+ey+f=0$.



$(x_0,y_0)\in C$, $\underline{y-y_0=t(x-x_0)}$

$0=f(x,y_0+t(x-x_0))=\underline{a(t)}x^2+\underline{b(t)}x+\underline{c(t)}$

$x^2+A(t)x+B(t)=0$.

$\left(x=-A(t)-x_0,\ y=t(x(t)-x_0)+y_0\right)$

Q: How do we know if a curve is rational or not?

- field of rational functions = function field

$k(X) = \left\{ u(x,y) = \frac{p(x,y)}{q(x,y)}\Big|_{C=X} : f \nmid q(x,y) \right\}$ field $\left(\overset{k=\mathbb{C}}{\mathbb{C}} \mathcal{M}(X) \right)$

$\frac{p_1}{q_1}\Big|_C = \frac{p_2}{q_2}\Big|_C \Leftrightarrow f \mid p_1 q_2 - p_2 q_1$

On $x^2+y^2=1$: $\frac{1-y}{x} = \frac{x}{1+y}$ $(0,1)$

$P \in C$, $u = \frac{p}{q}$ $q(P) \neq 0$, u is regular at P.

$x \to \varphi(t)$
$y \to \psi(t)$

Thm: $\{f(x,y)=0\} = X$ is rational $\Leftrightarrow$ $k(X) \cong k(t)$
$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\parallel$
$k(X) \hookrightarrow k(t)$ $\qquad\qquad$ $k(\text{a line})$

Pf: "$\Rightarrow$" $\exists\, \varphi(t), \psi(t)$, $f(\varphi(t), \psi(t)) \equiv 0$.

$k(X) \ni u(x,y) \longmapsto u(\varphi(t), \psi(t)) \in k(t)$. $\rightsquigarrow$ $k \subset k(X) \hookrightarrow k(t)$

$u(x,y) = \frac{p(x,y)}{q(x,y)}$

Thm (Lüroth) A subfield of $k(t)$ containing k is of the form $k(g(t))$.
1877 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong k(t)$

Lüroth Problem: Is a unirational subfield of $k(t_1, \cdots, t_n)$ containing k isomorphic to $k(t_1, \ldots, t_n)$?

dim 2: algebraic closed field char $= 0$ ✓

char $= p > 0$ ✗

dim ≥ 3: not true.

birational to $k^n \subset \mathbb{P}^n$

dim $= 4$: generic Cubic 4-folds are not rational

$$\underbrace{\text{Cubic 4-folds}}_{X = \{F(z_0, \ldots, z_4) = 0\} \subset \mathbb{P}^5} \qquad k(X) \Leftrightarrow k(t_1, \ldots, t_4).$$

"$\Leftarrow$" $k(X) \cong k(t)$ $\qquad f(x,y) = 0$.

$\Downarrow$ $\qquad\qquad\qquad\qquad \Downarrow$

$x \longmapsto \varphi(t)$ $\qquad f(\varphi(t), \psi(t)) = 0$

$y \longmapsto \psi(t)$.

$\frac{p(x,y)}{q(x,y)} = g(x,y) \longleftarrow t$

$\Rightarrow$ $x = \varphi(t)$, $y = \psi(t)$: $\qquad k \setminus \{t_1, \ldots, t_r\} \xrightarrow{1-to-1} X \setminus \{P_1, \ldots, P_s\}$.

$x = \varphi(g(x,y))$, $y = \psi(g(x,y))$. $\qquad \{q(x,y) = 0, f(x,y) = 0\}$

$y^2 = x^2 + x^3$, $\quad x = t^2 - 1$, $\quad y = t(t^2 - 1)$

$\qquad\qquad\qquad\quad t^4 - 1 \qquad\qquad t^2(t^4 - 1)$

- Rational Map. $\qquad k^2 = \text{plane}$

$\underbrace{u, v \in k(X)}_{= \frac{p_1}{q_1}}$, $\qquad \varphi(P) = (u(P), v(P)) \in k^2$

$\varphi: X \to Y$ rational map $\qquad \varphi(P) \in Y$ for every P at which φ is defined

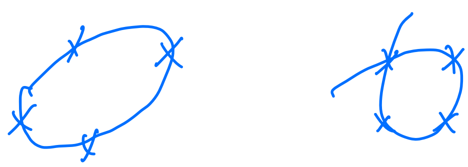
Birational map: $\psi: Y \to X$ s.t. $\varphi \circ \psi$, $\psi \circ \varphi$ both identity at points where they are defined.

$u \in k(X)$, $\psi^* u \in k(Y)$

$v \in k(Y)$, $\varphi^* v \in k(X)$ $\quad \Rightarrow \quad \boxed{k(X) \underset{\varphi^*}{\overset{\psi^*}{\rightleftarrows}} k(Y) \quad \cong}$

$\Downarrow$

Birational map: $\boxed{X \to Y}$



$x \mapsto \xi(u, v)$

$y \mapsto \eta(u, v)$

$\varphi(x, y) \mapsfrom u$

$\psi(x, y) \mapsfrom v$