Symmetric spaces: $\forall p \in M$, $\exists\ A_p: M \to M$ isometry $A_p(p) = p$

$dA_p = -Id: T_pM \to T_pM$

$\Updownarrow$ Cartan

$M = G/H$ · G Lie gp. with an involutive automorphism

$\sigma: G \to G$. $\sigma^2 = Id$.

· $(G^\sigma)^0 \subseteq H \subseteq G^\sigma$ closed sub gp.

$$\rightsquigarrow \begin{cases} \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \quad \sigma: \mathfrak{g} \to \mathfrak{g} \text{ Lie alg. automorphism.} \\ \qquad \sigma|_{\mathfrak{h}} = Id. \quad \sigma|_{\mathfrak{m}} = -Id. \\ \mathfrak{h} = \{X \in \text{isom} : X|_p = 0\} \\ \mathfrak{m} = \{X \in \text{isom} : \nabla X|_p = 0\}. \end{cases}$$

Ex:

Type I: $S^n = SO(n+1)/SO(n)$

$\updownarrow$

Type III: $H^n = SO(n,1)/S(O(n) \times \{1,-1\})$.

$\rightsquigarrow$ $G/H \updownarrow \tilde{G}/H$

Grassmannian $SO(k+l)/SO(k) \times SO(l)$

$\updownarrow$

$SO(k,l)/SO(k) \times SO(l)$

hyperbolic Grassmannian

Type I: $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\}) \Big/ \begin{smallmatrix}(z_1, \cdots, z_n) \\ \wr \\ (\lambda z_1, \cdots, \lambda z_n)\end{smallmatrix}\ \lambda \neq 0 = SU(n+1)/S(U(n) \times U(1))$

$\updownarrow \qquad \updownarrow$

Type III $\mathbb{CH}^n \cong \mathbb{B}^{2n} \subset \mathbb{C}^{2n} = SU(n,1)/S(U(n) \times U(1))$.

Type II:
Compact Lie gp.

$G = G\times G / \Delta_G$. $\Delta_G = \{(x,x) \mid x \in G\} \subset G\times G$

$\sigma : G\times G \to G\times G$ $\sigma : (x,y) \to (y,x)$. $\Delta_G = (G\times G)^\sigma$.

$\updownarrow$

Type IV: $\boxed{M = G^{\mathbb{C}}/G}$ Noncompact symmetric space.

Ex: $G = SU(n+1) = \{ A \mid (n+1)\times(n+1),\ A^*A = Id_{n+1},\ \det(A)=1\}$

$\| $

$G\times G/\Delta_G$

$\updownarrow$

$G^{\mathbb{C}}/G = SL(n+1,\mathbb{C})/SU(n+1) = \{$ Hermitian Inner products on $\mathbb{C}^{n+1}\}$

$\cong$

$\{[B]=[P\cdot U]\} = \{$ positive definite matrices of size $(n+1)\times(n+1)\}$

P: positive definite, U: unitary matrices.

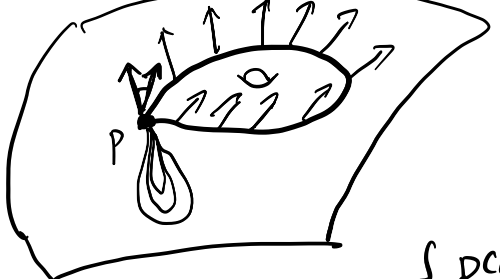
Type I & Type II are compact with $\sec \geq 0$.

Type III & Type IV are noncompact with $\sec \leq 0$.

$(R(X,Y)Y,X) = ([Y,[X,Y]],X)$ ($[X,Y]$ isom p)

$= \pm \|[X,Y]\|^2$.

<u>Holonomy</u> (M^n, g) Riem. mfd. c : loop $[0,1] \to M$
$c(0) = p = c(1)$



$P_{c(0)}^{c(1)} : T_pM \to T_pM$
$\cup$ $\quad \cong$
$O(n)$ $\quad \mathbb{R}^n$

$\{ P_{c(0)}^{c(1)} : c \text{ loop} \} = Hol(M,g)$
$\cap$
$O(n)$

$Hol^0(M,g) = \{ P_{c(0)}^{c(1)} : c \text{ contractible loop} \}$
restricted gp.

<u>Fact</u>: $\underline{Hol^0(M,g) \text{ is the identity component of } Hol(M,g)}$
which is a Lie subgp. of $O(n)$

$\rightsquigarrow$ $hol(M,g)$ = Lie algebra of $Hol^0(M,g) \subseteq so(n)$
$\boxed{\text{holonomy Lie algebra}}$ $\qquad so(n) = \{A : n \times n, \underline{A^T + A = 0}\}$.

<u>Thm</u> (Ambrose-Singer) $hol(M,g)$ is generated by curvature (mod. parallel transport)

$(M,g) \rightsquigarrow$ $R(X,Y) : T_pM \to T_pM \in so(n)$
$X, Y \in T_pM$ $\qquad z \mapsto R(X,Y)z$

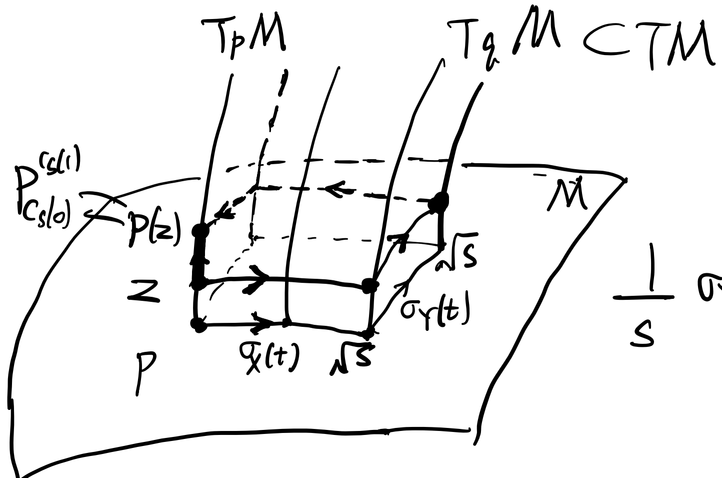
$(R(X,Y)z, w) = -(R(X,Y)w, z) \quad \forall z, w \in T_pM$

Fact: $\{R(X,Y) : X, Y \in T_pM\} \subseteq \mathfrak{hol}(M,g)$. (Cartan)

$$\mathrm{Hol} \ni \; \lim_{s\to 0} \frac{P^{c_s(1)}_{c_s(0)} - \mathrm{Id}}{s} = R(X,Y) \in T_e\mathrm{Hol}^\circ = \mathfrak{hol}(g).$$

In dim 2:

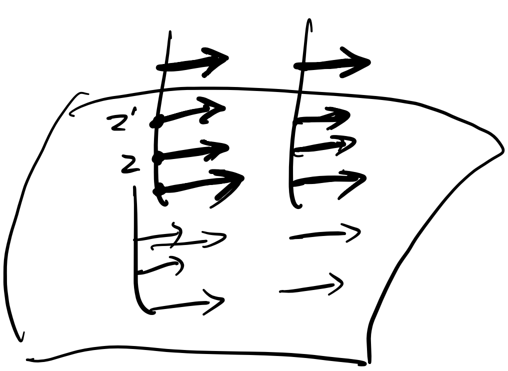
$$\frac{\theta(s)}{s} = \frac{2\pi - \sum_i \alpha_i(s)}{s} \xrightarrow{s\to 0} \text{Gauss curvature at } P$$



$$\frac{1}{s}\, \underbrace{\sigma_Y(\sqrt{s})^{-1}\sigma_X(\sqrt{s})^{-1}}_{\sigma_{Y^H}(\sqrt{s})}\, \underbrace{\sigma_Y(\sqrt{s})\,\sigma_X(\sqrt{s})}_{\sigma_{X^H}(\sqrt{s})}\, z$$

$$\Big\downarrow\; s \to 0$$

$$[Y^H, X^H]$$



$$X = \sum_i X^i \frac{\partial}{\partial x^i}$$

$$\rightsquigarrow X^H(z) = X^i \frac{\partial}{\partial x^i} - X^k \Gamma^i_{kj} z^j \frac{\partial}{\partial u^i}$$

$$z = z^i \frac{\partial}{\partial x^i}$$

Christoffel symbol

$$[X^H, Y^H] = [X^i\frac{\partial}{\partial x^i} - X^k\Gamma^i_{kj} z^j \frac{\partial}{\partial u^i}, Y^r\frac{\partial}{\partial x^r} - Y^s\Gamma^l_{rs} z^r \frac{\partial}{\partial u^l}]$$

$X^i(x)$ $\quad X^k(x)$ $\quad Y^r(x)$

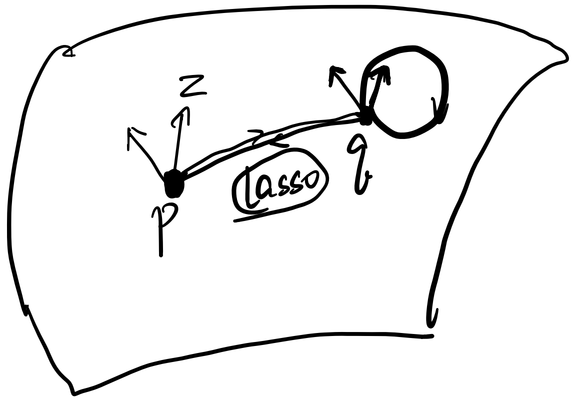
normal coord. $\Gamma(p)=0$

$\nabla Y|_p = 0$

$\nabla X|_p = 0$

$$= -X^i Y^s\left(\frac{\partial}{\partial x^i}\Gamma^l_{rs}\right) z^r \frac{\partial}{\partial u^l} + Y^r X^k\left(\frac{\partial}{\partial x^r}\Gamma^i_{kj}\right) z^j \frac{\partial}{\partial u^i}$$

$$= R(Y,X)z$$



$\forall q, \forall$ lasso

$$\left\{ \tau^p_q \cdot R(X_q, Y_q)\tau^q_p(z) \right\}$$

$\cap\!|$

$\mathfrak{hol}(M,g)$.

Thm (Ambrose–Singer) This is an identity.

Thm: for (irreducible) symmetric space ($\nabla R = 0$),

$$\boxed{\mathfrak{hol} = \mathfrak{isom}^0_p(M,g)}$$

$= \mathrm{Isom}(M,g) / \mathrm{Isom}_p(M,g)$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$

max{dim of flats}

$\max\{\dim \mathfrak{a} : \mathfrak{a} \subseteq \mathfrak{m}, [\mathfrak{a},\mathfrak{a}]=0\} = \mathrm{rank}(G/H)$.

Thm (Berger 1955) (Simons 1962) Let (M,g) be a simply connected irreducible Riem. n-mfd. Then either M is a symmetric space of rank ≥ 2

OR $\mathrm{Hol}(M,g) = \mathrm{Hol}^0(M,g)$ is one of the following group:

$\subset SO(n)$

dim $= n$	Hol	Properties	
n	$SO(n)$	Generic case	
$n=2m$	$U(m)$	Kähler	
$n=2m$	$SU(m)$	Kähler and $Ric = 0$	Calabi-Yau
$n=4m$	$Sp(m)$	HyperKähler	Conj: compact are symm.
$n=4m$	$Sp(1)\cdot Sp(m)$	Quaternion-Kähler, Einstein	
$n=16$	$Spin(9)$	Symmetric, Einstein	$\mathbb{OP}^2$
$n=8$	$Spin(7)$	$Ric=0$	
$n=7$	$G_2 = \mathrm{Aut}(\mathbb{O})$	$Ric=0$	M-theory, 11-dim $= 4+7$

metrics with exceptional holonomy

$\mathbb{O}$ = Octonion

CROSS: S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$.

(Compact Rank One Symmetric Space)

$\mathrm{Hol} = Sp(n) \times Sp(1)$.