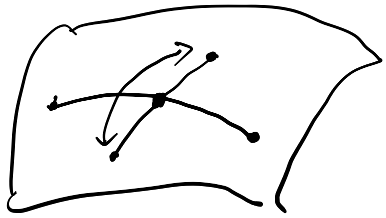
symmetric spaces: $\forall\, p \in M,\ \exists A_p : M \to M$. $A_p^* g = g$, $A_p^2 = Id$, $A_p(p) = p$, $dA_p = -Id : T_pM \to T_pM$

(M, g)



$$\Rightarrow \quad M = Isom(M,g)/Isom_p = Isom^0(M,g)/Isom_p$$

$$Isom_p = \{ f : M \to M : f^* g = g,\ f(p) = p \}$$

$$= G/K \qquad G = Isom^0(M,g), \quad K = Isom_p \cap Isom^0.$$

$$\leftrightsquigarrow \quad \mathfrak{g} = \text{Lie algebra of } G = t_p \oplus k_p . \quad \left\{ \begin{array}{l} \sigma : \mathfrak{g} \to \mathfrak{g} \text{ automorphism} \\ \sigma|_{t_p} = -Id,\ \sigma|_{k_p} = Id. \end{array} \right.$$

$$X \mapsto X_p$$

$$t_p \cong \{ X \in isom : \nabla X|_p = 0 \} \cong T_pM$$

$$k_p \cong isom_p = \text{Lie algebra of } Isom_p = \{ X \in isom \text{ (Killing)} : X|_p = 0 \}$$

$$\begin{pmatrix} X \\ \downarrow \\ \nabla X|_p \end{pmatrix} \qquad \downarrow$$

$$\{ A : T_pM \to T_pM, \text{ skew-sym. w.r.t. the inner product} \} = so(T_pM)$$

$$\Rightarrow \quad [t_p, t_p] \subseteq k_p \leftrightsquigarrow \boxed{[x, y] = \pm R(x, y) \in so(T_pM)}$$

$$[k_p, t_p] \subseteq t_p \leftrightsquigarrow [h, x] = h(x) \in t_p \cong T_pM$$

$$[k_p, k_p] \subseteq k_p \leftrightsquigarrow [h_1, h_2] = \pm (h_1 \circ h_2 - h_2 \circ h_1)$$

Curvature formula: $X, Y \in T_pM \rightsquigarrow$ Killing vector field

$Z \qquad X, Y, Z$ s.t. $\nabla X|_p = 0$, $\nabla Y|_p = 0$, $\nabla Z|_p = 0$.

1st Bianchi

$$R(X,Y)Z = -R(Y,Z)X + R(X,Z)Y$$

$$= \nabla^2_{Z,X}Y - \nabla^2_{Z,Y}X \quad \text{(formula true for Killing v.f.)}$$

$$= (\nabla_Z \nabla Y)(X) - (\nabla_Z \nabla X)(Y)$$

$$= \big(\nabla_Z(\nabla Y(X)) - \nabla Y(\nabla_Z X)\big) - \big(\nabla_Z(\nabla X(Y)) - (\nabla X)(\nabla_Z Y)\big).$$

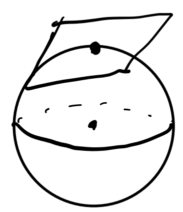
$$= \big(\nabla_Z \nabla_X Y - \nabla_{\nabla_Z X} Y\big)\big|_p - \big(\nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X\big)\big|_p$$

$$= \nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X = \nabla_Z [X,Y] - \underset{\substack{\| \\ 0}}{\nabla_{[X,Y]} Z}$$

$$= [Z,[X,Y]] = -[[X,Y],Z] = -\mathrm{ad}_{[X,Y]}(Z).$$

$$\Rightarrow \quad R(X,Y) = -\mathrm{ad}_{[X,Y]}.$$

Ex: $S^n = SO(n+1)/SO(n) = O(n+1)/SO \times \mathbb{Z}_2$



$\subset \mathbb{R}^{n+1}$

$O(n+1) \supseteq SO$

$SO \to O(n+1)$

$S \mapsto \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} S & 0 \\ 0 & -1 \end{pmatrix}$.

$\mathfrak{g} = so(n+1)$. $\quad \mathfrak{k} = so(n)$.

$\{A: (n+1)\times(n+1) \text{ matrix}, A^T = -A\} \hookleftarrow \{B: n\times n,\ B^T = -B\}$

$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \mapsfrom B$

$$t_p = \left\{ \begin{array}{c} \\ \left(\begin{array}{c|c} 0 & -x^T \\ \hline x & 0 \end{array}\right) \\ \underbrace{\ }_{n}\ \underbrace{\ }_{1} \end{array} \begin{array}{l} \}n \\ \}1 \end{array} \right\} \subseteq \mathfrak{g}. \qquad \mathfrak{g} = t_p \oplus \mathfrak{k}.$$

$\boxed{[t_p, t_p] \subseteq k_p}$. $\qquad X, Y \in t_p$, $X \leftrightarrow x$, $Y \leftrightarrow y$

$2(x_1y_1 + \cdots + x_n y_n) = 2\,\mathrm{tr}(x \cdot y^T) =$

$$\mathrm{tr}(X \cdot Y^T) = \mathrm{tr}\left(\begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & y^T \\ -y & 0 \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} x^T y & 0 \\ 0 & x \cdot y^T \end{pmatrix} = \mathrm{tr}(x^T y) + \mathrm{tr}(x \cdot y^T)$$

$\mathrm{tr}(x^T y) = \mathrm{tr}(y \cdot x^T) = \mathrm{tr}(x \cdot y^T)$

$$R(X,Y)Z = [Z,[X,Y]] \qquad Z \leftrightarrow \begin{pmatrix} 0 & -z^T \\ z & 0 \end{pmatrix}.$$

$$[X,Y] = \left[\begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y^T \\ y & 0 \end{pmatrix} \right] \qquad \begin{matrix} x=(x_1 \cdots x_n) \\ y=(y_1 \cdots y_n) \end{matrix}$$

$$= \begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & -y^T \\ y & 0 \end{pmatrix} - \begin{pmatrix} 0 & -y^T \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix} \qquad y^T\big(\ \big)\big(x\big) - x^T\big(\ \big)\big(y\big)$$

$$= \begin{pmatrix} -x^T y & 0 \\ 0 & -x y^T \end{pmatrix} - \begin{pmatrix} -y^T x & 0 \\ 0 & -y x^T \end{pmatrix} = \begin{pmatrix} y^T x - x^T y & 0 \\ 0 & \overset{0}{y x^T - x y^T} \end{pmatrix}$$

$$(y_1 \cdots y_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - (x_1 \cdots x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} y^T x - x^T y & 0 \\ 0 & 0 \end{pmatrix} \in k_p = so(n) \subset so(n+1)$$

$$[z,[x,y]] = \begin{pmatrix} 0 & -z^T \\ z & 0 \end{pmatrix}\begin{pmatrix} y^T x - x^T y & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} y^T x - x^T y & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -z^T \\ z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ z(y^T x - x^T y) & 0 \end{pmatrix} - \begin{pmatrix} 0 & -(y^T x - x^T y) z^T \\ 0 & 0 \end{pmatrix}$$

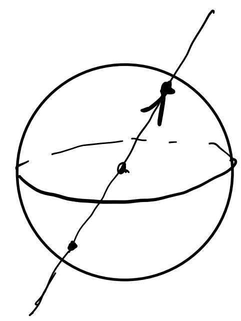
$$= \begin{pmatrix} 0 & \overbrace{y^T x \cdot z^T - x^T y z^T}^{= -(z \cdot y^T x - z \cdot x^T y)^T} \\ \underbrace{z \cdot y^T x - z \cdot x^T \cdot y}_{= (z,y)x - (z,x)y} & 0 \end{pmatrix} = R(X,Y)Z$$

$$\Rightarrow \quad R(x,y)z = (z,y)x - (z,x)y$$

$$\Rightarrow \quad (R(x,y)y,x) = ((y,y)x - (y,x)y, x) = \underline{(y,y)(x,x) - (x,y)^2}$$

$$\Rightarrow \quad sec(x,y) = 1 \quad \text{constant}.$$

$$S^n = \widetilde{Gr}(1, \mathbb{R}^{n+1}) = \{\underline{\text{oriented}} \text{ lines in } \mathbb{R}^{n+1}\}$$



generalize

oriented Grassmannian.

$$M = \widetilde{Gr}(k, \mathbb{R}^{n+1}) = \{\text{oriented } k\text{-dim. subspace in } \mathbb{R}^{n+1}\}$$

$$= O(n+1) / SO(k) \times O(n+1-k).$$

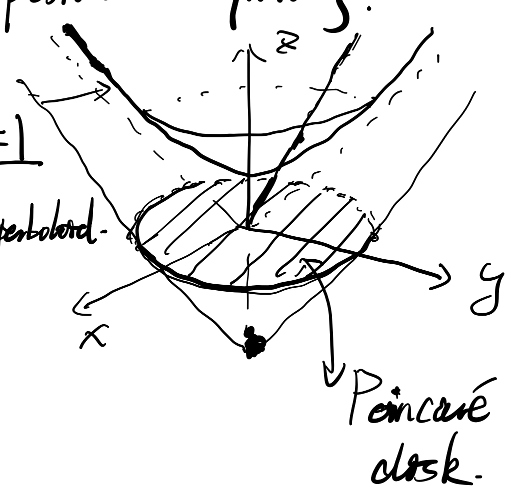
Ex: $\mathbb{H}^n$ = hyperbolic space $= SO(n,1)/SO(n)$

$$SO(n,1) = \left\{ A : n\times n \text{ matrix},\ A^T \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\mathbb{H}^n = \{ \ell \in \mathbb{R}^{n+1} : q|_\ell \text{ is positive definite} \}.$$

$z^2 - x^2 - y^2 = 1$

$= \boxed{SO(n,1)/SO(n)}$ = "sphere" = hyperboloid.



Ex: G compact Lie gp. e.g. $(S^1)^r$

$SU(2) \cong S^3$.

$\downarrow \qquad \downarrow$

$SO(3) \cong \mathbb{R}P^3$

$$\boxed{G \times G / \Delta_G} = G \qquad (G\times G)^\sigma$$

$$\Delta_G = \{(g,g) : g \in G\} \hookrightarrow G \times G.$$

$$\boxed{\begin{aligned}\sigma: G\times G &\to G\times G\\ (g_1,g_2) &\to (g_2,g_1)\end{aligned}}$$

$$G\times G/\Delta_G \xrightarrow{\cong} G \text{ diffeomorphism}.$$

$$(g_1, g_2) \longmapsto g_1 \cdot g_2^{-1}$$

$$(g_1,g_2) \ni \mathfrak{g} \text{ (Lie alg. of } G)$$

$$\boxed{\begin{aligned}\sigma: \mathfrak{g} \oplus \mathfrak{g} &\longrightarrow \mathfrak{g} \oplus \mathfrak{g}\\ (X, Y) &\to (Y, X)\end{aligned}}$$

Eigenspace for 1: $\underline{k_p = \mathrm{Lie}\,\Delta_G} = \{(X,X) : X \in \mathfrak{g}\}$

$-1: \{(X,-X),\ X \in \mathfrak{g}\} \cong \mathfrak{g} = T_eG \cong T_pM$

$$R(X,Y)Z = \underline{R((X,-X),(Y,-Y))(Z,-Z)}$$
$$= [(Z,-Z),[(X,-X),(Y,-Y)]]$$
$$= ([Z,[X,Y]], -[Z,[X,Y]]).$$

$$\boxed{(g,g)\cdot(X,-X) = (\mathrm{Ad}_g X, -\mathrm{Ad}_g X).\quad K_p \text{ action}}$$

$A_e : G \longrightarrow G$ $\rightsquigarrow$ $A_x : G \xrightarrow{A_x} G$

$G/\{e\}$ $/\!/$ $g \longmapsto g^{-1}$

not symmetric space data

$$\begin{array}{ccc} G & \xrightarrow{A_x} & G \\ \downarrow & & \downarrow \\ G & \xrightarrow{A_e} & G \end{array}$$

$g \mapsto g^{-1}$ $\quad \sigma(g_1 g_2)$

$(g_1, g_2) \mapsto (g_1 \cdot g_2)^{-1} = g_2^{-1} g_1^{-1}$ not a automorphism.

$\not\parallel$

$\sigma(g_1) \cdot \sigma(g_2)$

$L_g : G \to G$
$g_1 \mapsto g \cdot g_1$

$R_g : G \to G$
$g_1 \mapsto g_1 \cdot g$

Compact Lie gp. G

- Any inner product of $T_e G = \mathfrak{g}$

$\rightsquigarrow$ left-invariant Riem. metric on G

- Inner product of $T_e G = \mathfrak{g}$ invariant under adjoint action: $\underset{G \times \mathfrak{g}}{g \cdot x} = \frac{d}{dt}\Big|_{t=0} g \cdot \exp(t \cdot x) \cdot g^{-1}$

$\qquad \qquad = d \cdot Ad_g : T_e G \to T_e G$.

$\rightsquigarrow$ $\boxed{\text{bi-invariant Riem. metric on } G}$

$\Longleftrightarrow$ left and right multiplications are isometries.

$$\boxed{M = G/H}$$

$$\sigma : \underset{\substack{\Downarrow \\ g}}{\overset{\substack{\mathrm{Isom}(M,g) \\ \|}}{G}} \underset{\mapsto}{\rightarrow} \underset{A_p \circ g \circ A_p}{G}$$

involutive automorphism. $\sigma^2 = Id$

$$\sigma(e) = A_p \circ A_p = Id = e$$

$$\sigma(g_1 \cdot g_2) = A_p \circ g_1 g_2 \circ A_p = A_p \circ g_1 \circ A_p \circ A_p \circ g_2 \circ A_p$$

$$= \sigma(g_1) \cdot \sigma(g_2).$$

$$\sigma^2(g) = A_p \circ A_p \circ g \circ A_p \circ A_p = g.$$

$$(G^\sigma)^0 \subseteq K \subseteq G^\sigma.$$

$$K \to SO(T_pM)$$

$$\Updownarrow$$

isometry

$$\mathrm{Isom}_p$$

$$\|$$

$$\cdot \quad \mathfrak{g} = \mathfrak{t}_p \oplus \mathfrak{k}. \qquad K \circlearrowright \mathfrak{t}_p$$

$$\|\wr$$

$$T_pM$$

$$\Updownarrow$$

$$\boxed{\mathfrak{k} \hookrightarrow so(T_pM)}$$