- (M, g) Symmetric space:

$\forall p \in M$, $\exists$ an isometry $A_p : M \to M$ s.t. $\cdot\ A_p(p) = p$

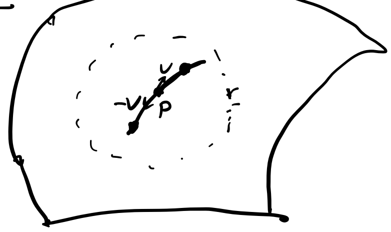
$\cdot\ dA_p = -Id_{T_pM} : T_pM \to T_pM$

$\Rightarrow$ $\nabla R = 0$ $\leftarrow$ locally symmetric

$\Downarrow$

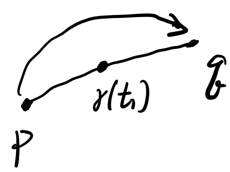
$\forall p \in M$, $\exists$ local geodesic symmetry

$$A_p(\exp_p(tv)) = \exp_p(t\cdot(-v))$$



M simply connected
$\underset{\text{complete}}{\Longrightarrow}$ global symmetric space.

Symmetric space $\Longrightarrow$ M is homogeneous



$$M = \text{Isom}(M) / \text{Isom}_p(M) \longrightarrow M$$

$$[g] \longmapsto g \cdot p = q$$

$$[g] = [g'] = [g(g^{-1}g')] \Leftarrow g^{-1}g' \in \text{Isom}_p(M) \Leftarrow (g^{-1}g')p = p \Leftrightarrow q = g' \cdot p$$

G Lie gp

G/H is a symmetric space $\Longleftrightarrow$ $\exists\ \sigma : G \to G$ involutive automorph.

$(G^\sigma)^0 \subseteq H \subseteq G^\sigma$

H: closed subgp.

- Killing vector field $=$ isom$(M) = \{ X$ sm. vector field. $\mathcal{L}_X g = 0 \}$

$\Updownarrow$

$\varphi(t)^* g = g$

$\varphi(t) = \exp(tX)$, i.e. $(\exp(tX))^* g = g$

X is Killing $\Longleftrightarrow$ ∇X is skew symmetric

$$\nabla X : TM \to TM$$
$$Y \mapsto \nabla_Y X$$

$$(\nabla_Y X, Z) = -(Y, \nabla_Z X) \quad \forall\, Y, Z \in TM$$

Fix $p \in M$

$$\mathfrak{isom} \longrightarrow T_pM \oplus \underline{so(T_pM)} \qquad \underline{\text{injective}}$$
$$X \longmapsto (X(p),\ \nabla X|_p)$$

<u>Fact</u>: killing vector field is Jacobi along any geodesics.

$$\cdot \quad M = \underline{Isom}/Isom_p .$$

$$\mathfrak{isom} \longrightarrow \underbrace{T_pM}_{\mathfrak{t}_p} \oplus \underbrace{\mathfrak{isom}_p}_{\mathfrak{k}_p} = \{\nabla X|_p : X \in \mathfrak{isom}\} \subseteq so(T_pM)$$

$$\underline{X \longmapsto (X(p),\ \nabla X|_p)}$$

$$\mathfrak{g} = \mathfrak{t}_p \oplus \mathfrak{k}_p$$

$\uparrow$ sub-Lie algebra

$$X \longleftrightarrow \frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot q$$

$$\mathfrak{t}_p = \underline{\{X \in \mathfrak{isom} : \nabla X|_p = 0\}} \ (\rightsquigarrow \text{translation})$$

$$\mathfrak{isom}_p = \mathfrak{k}_p = \underline{\{X \in \mathfrak{isom} : X|_p = 0\}} \ (\rightsquigarrow \text{rotation})$$

<u>Lem</u>: $\underline{[\mathfrak{t}_p, \mathfrak{t}_p] \subseteq \mathfrak{k}_p} = \mathfrak{isom}_p$

$[\mathfrak{t}_p, \mathfrak{k}_p] \subseteq \mathfrak{t}_p$

$[\mathfrak{k}_p, \mathfrak{k}_p] \subseteq \mathfrak{k}_p = \underline{\mathfrak{isom}_p}$

<u>Pf</u>: $\cdot\ X, Y \in \mathfrak{t}_p$: $\nabla X|_p = 0 = \nabla Y|_p$

$$[X,Y]|_p = \underline{\nabla_X Y} - \underline{\nabla_Y X}|_p = \underline{0} \implies [X,Y] \in \mathfrak{k}_p .$$

- $X \in \mathfrak{t}_p$, $\underline{Y \in \mathfrak{k}_p} \Rightarrow \nabla X|_p = 0$, $Y|_p = 0$

$$[X,Y]|_p = \nabla_X Y - \underline{\nabla_Y X}|_p = \nabla_X Y = \underbrace{(\nabla Y)}_{\in\, so(T_pM)}(X) \in T_pM \cong \mathfrak{t}_p.$$

Pick any $Z_p \in T_pM \rightsquigarrow$ Killing vector field $\underset{\in\, \mathfrak{t}_p}{Z}$ on M. $\nabla Z|_p = 0$

$$\nabla_Z [X,Y] \overset{\nabla Z|_p = 0}{=} [Z,[X,Y]]$$

curvature formula $\rightarrow = R(X,Y)Z = 0$ since $Y_p = 0$.

$\Rightarrow [X,Y] \in \mathfrak{t}_p$

$$[X,Y]|_p = (\nabla Y)(X) \in T_pM.$$

$$\mathfrak{g} = \underset{\ni x}{\mathfrak{t}_p} \oplus \underset{\ni h}{\mathrm{isom}_p} \qquad \mathrm{isom}_p \subseteq \underline{so(T_pM)} = \{h : T_pM \to T_pM\}$$

$$[h_1,h_2] = -(h_1 \circ h_2 - h_2 \circ h_1)$$

$$[h,x] = h(x)$$

$[x,y] \in \mathrm{isom}_p \rightsquigarrow$ curvature of symmetric space.

Formula for

- Curvature tensor of symmetric spaces.

Need Fact:

For killing vector field $K = \sum_i K^i \partial_i$ under local coordinate

$$\nabla^2_{X,Y} K = -R(K,X)Y \longleftrightarrow K^i{}_{,jk}\,\partial_i$$

$$\nabla^2_{X,Y}K = \nabla^2 K(X,Y) = (\nabla_X \nabla K)(Y) = \nabla_X \nabla_Y K - \nabla_{\nabla_X Y} K$$

$T^*M \otimes T^*M \otimes TM \ni \nabla^2 K$

$$K^i{}_{,jk}\,\partial_i = -R(K^i\partial_i, \partial_j)\partial_k = -K^i R_{ijk}{}^l \partial_l$$

Bianchi

$$R(X,Y)Z = -R(Y,Z)X + R(X,Z)Y$$

$$= \nabla^2_{Z,X} Y + (-\nabla^2_{Z,Y} X)$$

$$\boxed{T_pM \cong t_p = \{X : \nabla X|_p = 0\}}$$

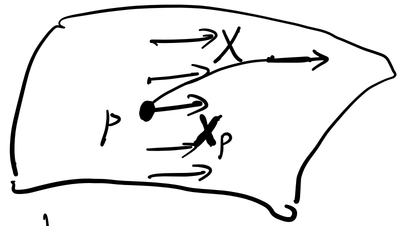
$$= (\nabla_Z \nabla Y)(X) - (\nabla_Z \nabla X)(Y)$$

$$= (\nabla_Z \nabla_X Y - \underbrace{\nabla_{\nabla_Z X} Y}_{=0 \text{ at } p}) - (\nabla_Z \nabla_Y X - \underbrace{\nabla_{\nabla_Z Y} X}_{=0 \text{ at } p})$$

$$= \nabla_Z(\nabla_X Y - \nabla_Y X) = \nabla_Z [X,Y] - \nabla_{[X,Y]} Z + \nabla_{[X,Y]} Z$$

$$\overset{\text{at } p}{=} [Z,[X,Y]] = -[[X,Y],Z].$$

$$R(X,Y)Z = -\underline{[[X,Y],Z]}$$



$$T_pM \longrightarrow \text{Isom}$$

$$\begin{array}{ccc} \cup\!\!\!\mid & & \cup\!\!\!\mid \\ X_p & \longmapsto & X \end{array} : \text{killing} \quad X|_p = X_p$$

$$\underline{\nabla X|_p = 0}$$

$K^i \partial_i = K$ killing $\iff$ ∇K skew-symmetric

$$\iff \boxed{K_{i,j} = -K_{j,i}} \qquad \{\partial_i\} \text{ o.n.b. at } p. \quad \partial_i = \frac{\partial}{\partial x^i}$$

$$\begin{array}{ccc} K_{i,j} & & -K_{j,i} \\ \| & & \| \\ (\nabla K(\partial_j), \partial_i) & & -(\partial_j, \nabla_{\partial_j} K) \end{array}$$

$$K_{i,jk} = -K_{j,ik} = -(K_{j,ki} + \underline{K_l R_{ljki}})$$

$$= K_{k,ji} - K_l R_{ljki}$$

$$= K_{k,ij} + K_l R_{lkij} - K_l R_{ljki}$$

$$= -K_{i,kj} + \text{— — — .}$$

$$= -(K_{i,jk} + K_l R_{lijk}) + K_l R_{lkij} - K_l R_{ljki}$$

$$\Rightarrow 2K_{i,jk} = -K_\ell R_{\ell ijk} + K_\ell R_{\ell kij} - K_\ell R_{\ell jki}$$

$$= -K_\ell (R_{\ell ijk} + R_{\ell jki}) + K_\ell R_{\ell kij}$$

$$-K_\ell (R_{\ell ijk} + R_{\ell jki}) = K_\ell \cdot R_{\ell kij}, \qquad K_\ell R_{\ell kij} = 2 \cdot K_\ell R_{\ell kij}$$

$$\Rightarrow K_{i,jk} = K_\ell \cdot R_{\ell kij} = -K_\ell R_{\ell kji}$$

for any killing vector field

$$(\nabla^2 K)(\partial_k, \partial_j) = -R(K, \partial_k)\partial_j$$

$$\nabla^2_{\partial_k, \partial_j} K = (\nabla_{\partial_k} \nabla K)(\partial_j)$$

$$\Longleftrightarrow (\nabla_X \nabla K)(Y) = -R(K, X)Y, \ \forall X, Y \in TM$$