Killing Vector Fields: X is killing if $\mathcal{L}_X g = 0$.

$$\mathcal{L}_X g = \frac{d}{dt}\Big|_{t=0} \sigma(t)^* g$$

$\sigma(t) = \exp(tX)$: 1 psg. of transformations of X generated by X.

killing $\iff$ $\sigma(t)$ is isometry of (M, g) (for $|t| \ll 1$).

$$\boxed{\underset{\text{isom}}{\overset{\shortparallel}{\text{isom}(M,g)}} = \{\text{Killing vector fields on } M\}}$$

Lemma: X is killing $\iff$ $\nabla X : TM \to TM$, $Y \mapsto \nabla_Y X$ is skew symmetric.

$(\nabla_Y X, Z) = -(Y, \nabla_Z X)$ $\forall\, Y, Z \in TM$.

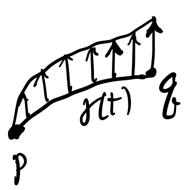
$\Updownarrow$

$\left(A^T = -A \text{ under any o.n.b.} \right)$

Prop:
$$\text{isom} \longrightarrow T_pM \times so(T_pM)$$
$$X \longmapsto (X(p), \nabla X|_p)$$
is an injective linear transformation.

Pf: Show kernel $= 0$. Assume $X(p) = \nabla X(p) = 0$.

- $X|_\gamma$ is a Jacobi field.



$$\Rightarrow \begin{cases} \boxed{X'' + R(\gamma', X')\gamma' = 0} \\ X(0) = 0 \\ X'(0) = 0 \end{cases} \Longrightarrow X \equiv 0 \text{ along } \gamma.$$

- X killing $\Longrightarrow$ $\overset{\exp(sX)}{\overset{\shortparallel}{\sigma(s)}}$ isometry, $|s| \ll 1$. $\Rightarrow$ $\sigma(s) \circ \gamma$ is also geod. $\forall\, |s| \ll 1$.

$X|_\gamma = \frac{d}{ds}\Big|_{s=0} (\sigma(s) \circ \gamma)$ is a Jacobi field
$\shortparallel$
vector field generating geodesic variations. ∎

Cor: $\dim_{\mathbb{R}} \mathrm{isom}(M,g) \leq \underline{n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}}$

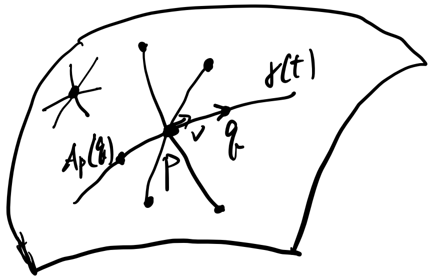
Fact: "=" holds iff (M,g) has constant curvature.

e.g. S^n, $\mathbb{R}^n$, $\mathbb{H}^n$

$\mathrm{isom}\,\mathbb{R}^n = \mathrm{Span}\left\{\begin{matrix}\text{translations} \\ \text{rotations}\end{matrix}\right\}$

Symmetric Spaces

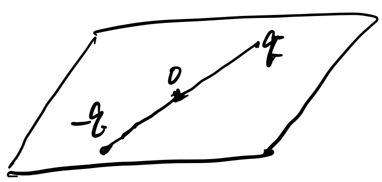
Def: A Riem. mfd. (M,g) is a symmetric space if for any point $p \in M$, there is an isometry $A_p: M \to M$ such that $A_p(p) = p$ and $dA_p: T_pM \to T_pM$ is $\underline{-Id_{T_pM}}$.



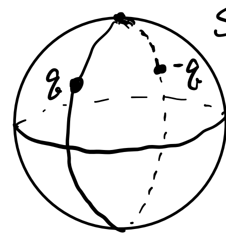
$$\rightsquigarrow A_p(\underbrace{\exp_p(t\cdot v)}_{\gamma(t)}) = \exp_p(t\cdot(-v))$$

geodesic symmetry at p.

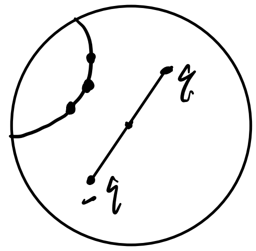
Ex: $\mathbb{R}^n$



S^n



$\mathbb{H}^n$



Thm (Cartan): (M,g) symmetric space, then $\boxed{\nabla R = 0}$

R: $(3,1)$ tensor

$(X,Y,Z) \to R(X,Y)Z \in TM$.

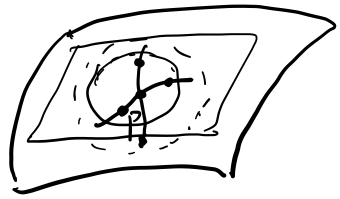
Pf: ∇R: $(4,1)$ tensor

$dA_p: T_pM \to T_pM$ vs $dA_p(\nabla R) \overset{dA_p = -Id}{=} \ominus \nabla R \Rightarrow \nabla R = 0$. ∎

$\underset{\nabla R}{\overset{\shortparallel}{\,}} \leftarrow A_p$ isometry.

(Cartan)

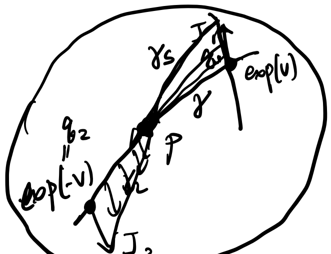
Thm: If $\nabla R = 0$ on (M,g), then for any $p \in M$, there is a geodesic isometry in a nbhd. of $p \in M$.



$$\begin{array}{ccc} B_\delta(0) & \xrightarrow{v \mapsto -v} & B_\delta(0) \\ \wr\!\shortparallel \exp_p & & \wr\!\shortparallel \exp_p \\ A_p: U & \longrightarrow & U \end{array}$$

$\exp(tv) \mapsto \exp(-tv)$.

Pf:



$U \simeq B_p(0) \subset T_pM$.

$\exp(v) \longleftarrow v$

$$\frac{d}{ds}\Big|_{s=0} \underbrace{\left(\exp_p(t(v+sw_1)\right)}_{\gamma_s(t)}$$

$J_1'' + R(\gamma', J_1)\gamma' = 0$

$$\boxed{\underset{J_1(1)}{J_1 = (d\exp_p)_v(w_1)} \quad J_1(t) = (d\exp_p)_{tv}(tw_1),}$$

$$dA_p \cdot J_1^{(t)} = \frac{d}{ds}\Big|_{s=0} A_p(\gamma_s(t)) = \frac{d}{ds}\Big|_{s=0} \exp_p(-t(v+sw_1)).$$

$dA_p \cdot J_1^{(t)} = dA_p \cdot \dfrac{d}{ds}\gamma_s(t)\Big|_{s=0}$

$\dfrac{d}{ds}\Big|_{s=0} \exp_p(-t(v+sw_1)) = (d\exp_p)_{-tv}(-tw_1) = J_2(t)$.

$J_1'' + R(\gamma_1', J_1)\gamma_1' = 0$: ∘ Choose <u>o.n.b.</u> $\{e_i\}$ for T_pM

$\rightsquigarrow$ ∘ parallel transport $\{e_i(t)\}$ along γ

$J = \sum_i a_i(t)\, e_i(t)$, $e_1(t) = \gamma'(t)$

$|J|^2 = \sum_i |a_i(t)|^2$

$\rightsquigarrow \sum_i a_i'' \cdot e_i(t) + \underbrace{R(e_1, \sum_j a_j \cdot e_j)\, e_1}_{\displaystyle \sum_j (R(e_1, e_j)e_1, e_i)\, a_j e_i = \sum_j R_{1j1i}(t) \cdot a_j \cdot e_i} = 0$

$\{a_i'' + \sum_j R_{1j1i}\, a_j = 0\} \Longleftrightarrow$

$\boxed{\underline{\nabla R = 0} \rightsquigarrow \underline{R_{1j1i} \text{ independent}} \text{ of } t.}$

$\underline{J_2'' + R(\gamma_2', J_2)\gamma_2' = 0}$ $\rightsquigarrow$ $J_2 = \sum_i b_i(t)\, e_i(t)$

$\rightsquigarrow b_i(t) = -a_i(t)$

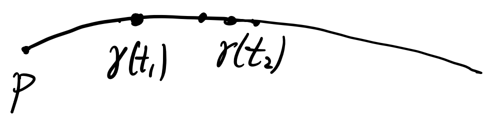
$\Rightarrow |J_1|^2 = \sum_i |a_i(t)|^2 = \sum_i |b_i(t)|^2 = |J_2(t)|^2$ ∎

<u>Def</u>: (M, g) is locally symmetric if $\nabla R \equiv 0$.

<u>Fact</u>: $\left.\begin{array}{l} M \text{ locally symmetric} \\ \text{simply connected} \\ \text{complete} \end{array}\right\} \Longrightarrow M$ symmetric

Fact: M symmetric $\Longrightarrow$ M is geodesic complete

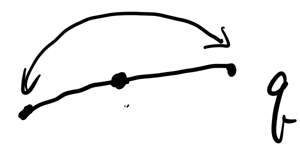
$\Longrightarrow$ M is homogeneous.



$$\underbrace{A_{\gamma(t_2)} \circ A_{\gamma(t_1)}(\gamma(t))} = A_{\gamma(t_2)} \circ \gamma(2t_1 - t)$$

$$= \gamma(2t_2 - (2t_1 - t))$$

$$= \gamma(2(t_2 - t_1) + t)$$



homogeneous.

Thm (Cartan) M is symmetric space ($\exists$ geodesic isometry at any pt.) $\boxed{\Longrightarrow \nabla R = 0}$ Locally symmetric

$\Updownarrow$

$\boxed{\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}}$ $M \cong G/H$ where G is a connected Lie gp.

$\downarrow$ $\updownarrow$

$\boxed{so(T_pM) \oplus T_pM}$

H is a compact subgp. of G

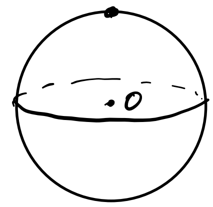
s.t. $\exists$ an involutive ($\sigma^2 = Id_G$) automorphism (diffeomorphism that preserves the gp. str) $\sigma : G \to G$

for which $(G^\sigma)^0 \subset H \subset G^\sigma$.

$(G^\sigma)^0$: connected component of $1 \in G^\sigma$

$G^\sigma = \{x \in G : \sigma(x) = x\}$

Ex: $S^n = SO(n+1)/SO(n)$



$\subset \mathbb{R}^{n+1}$

$\mathbb{RP}^n = O(n+1)/O(n)\times\{1,-1\}$

$\mathbb{H}^n = SO(n,1)/SO(n)$