Thm: (M^n, g) closed Riem. mfd. (orientable) $\mathcal{R}_g : \Lambda^2 TM \to \Lambda^2 TM$ (symmetric linear transform)

If $\mathcal{R}_g \geq 0$, then any harmonic form is parallel.

If $\mathcal{R}_g > 0$, then any harmonic p-form vanish for $p = 1, \cdots, n-1$.

$\Downarrow$

$(b_p(M) = 0, \ 1 \leq p \leq n-1)$

Pf: Derive the Weitzenböck formula: $\forall \omega \in \Omega^p(X)$

$\{e_i\}$ o.n. frames

$\{\theta^i\}$ dual basis

$$\Delta \omega = \nabla^* \nabla \omega + \frac{1}{4} \sum_{i,j} [\theta^i \cdot \theta^j \cdot, R(e_i, e_j)\omega]$$

Clifford multiplication.

$$(\Delta\omega, \omega)_g = (\nabla^*\nabla\omega, \omega) + \frac{1}{4} \sum \big(\underbrace{[\theta^i \cdot \theta^j, R(e_i, e_j)\omega]}_{= \, ad_{\theta^i \cdot \theta^j}(R(e_i,e_j)\omega)}, \omega \big)_g$$

$$\| \quad -\frac{1}{4} \sum_{i,j} \big(R(e_i, e_j)\omega, \, ad_{\theta^i \cdot \theta^j}(\omega) \big)_g$$

$$\| \quad -\frac{1}{4} \big((R(e_i, e_j)e_k, e_l) \, ad_{\theta^k \cdot \theta^l}(\omega), \, ad_{\theta^i \cdot \theta^j}(\omega) \big)_g$$

$$\| \quad \frac{1}{4} \big((R(e_i \wedge e_j), e_k \wedge e_l)_g \, ad_{\theta^k \cdot \theta^l}(\omega), \, ad_{\theta^i \cdot \theta^j}(\omega) \big)_g$$

$$\| \quad \frac{1}{4} \sum_{\alpha=1}^{\binom{n}{2}} \lambda_\alpha \| [\Theta_\alpha, \omega] \|^2$$

$\{e_i \wedge e_j\}$ o.n.b. for $\Lambda^2 TM$ $\rightsquigarrow$ o.n.b. for $\Lambda^2 TM$ s.t. $R(\omega_\alpha) = \lambda_\alpha \omega_\alpha$.

$\{\theta^i \wedge \theta^j\}$ dual basis for $\Lambda^2 T^*M$ and dual basis $\{\Theta_\alpha\}$

$$\big((R(X \wedge Y), Z \wedge W)_g = (R(X,Y)W, Z) = -(R(X,Y)Z, W)_g \big)$$

$$R \geq 0 \iff \lambda_\alpha \geq 0 \quad \alpha = 1, \cdots, \binom{n}{2}.$$

$$(>0) \qquad (\lambda_\alpha > 0)$$

$$\underbrace{\int_M (\Delta\omega,\omega)\,dvol}_{=\,0} = \int_M (\nabla^*\nabla\omega,\omega)\,dvol + \frac{1}{4}\int_M \sum_\alpha \lambda_\alpha \|[\Theta_\alpha,\omega]\|^2$$

ω harmonic

$$\int_M (\nabla^*\nabla\omega,\omega)\,dvol = \int_M \|\nabla\omega\|^2\,dvol = (\nabla\omega,\nabla\omega)$$

$$\lambda_\alpha \geq 0 \ \forall\alpha \Rightarrow \nabla\omega \equiv 0. \quad \sum_\alpha \lambda_\alpha \|[\Theta_\alpha,\omega]\|^2 \equiv 0.$$

ω parallel

$$\lambda_\alpha > 0 \ \forall\alpha \Rightarrow \nabla\omega = 0, \quad [\Theta_\alpha,\omega] \equiv 0 \ \forall\alpha \quad \left(\{\Theta_\alpha\} \text{ basis for } \Lambda^2 T^*M\right)$$

$$\Downarrow$$

$$[\theta^i \bullet \theta^j, \omega] \equiv 0 \quad \forall\, 1 \leq i < j \leq n.$$

$$\Downarrow \quad p\text{-form with } 1 \leq p \leq n-1$$

$$\omega \equiv 0.$$

$$\boxed{\begin{aligned} \omega \bullet \theta^j &= (-1)^p(\theta^j \wedge \omega + \iota_{e_j}\omega) \\ \theta^j \bullet \omega &= \theta^j \wedge \omega - \iota_{e_j}\omega \end{aligned}}$$

$$\boxed{\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p}\, \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}}$$

$$[\theta^i \bullet \theta^j, \overbrace{\theta^{i_1} \wedge \cdots \wedge \theta^{i_p}}^{\theta^I}] = \begin{cases} 0 & i,j \notin \{i_1,\ldots,i_p\} = I \\ 0 & i,j \in \{i_1,\ldots,i_p\} \\ 2\theta^i \bullet \theta^j \bullet \theta^I & \text{otherwise} \quad (i \in I \text{ and } j \notin I \text{ or } j \in I \text{ and } i \notin I) \end{cases}$$

$$2\theta^i \bullet \theta^j \bullet \theta^I = \pm 2 \cdot \theta^j \wedge \theta^{I \setminus \{i\}}$$

$$\left(\begin{aligned} & \iota_{e_j}\theta^i = \delta_{ij} \qquad \iota_{e_j}(\omega \wedge \eta) = (\iota_{e_j}\omega) \wedge \eta + (-1)^{\deg\omega}\, \omega \wedge \iota_{e_j}\eta \\ & \underbrace{(\iota_v\omega)}_{(p-1)\text{-form}}(v_1,\ldots,v_{p-1}) = \omega(v, v_1, \ldots, v_{p-1}) \end{aligned} \right)$$

$$\theta^i \bullet \theta^j \bullet \theta^I = \theta^i \bullet (\theta^j \wedge \theta^I - \iota_{e_j}\theta^I) = \theta^i \bullet \theta^j \wedge \theta^I$$

$$= \underbrace{\theta^i \wedge \theta^j \wedge \theta^I}_{=\,0} - \iota_{e_i}(\theta^j \wedge \theta^I)$$

Special case: • If $\underline{Ric \geq 0}$. Then harmonic 1-form is parallel

$Ric > 0$ harmonic 1-form $\underline{\text{vanish}}$ $\Rightarrow \underline{b_1(M) = 0}$.

• $\underline{\text{Thm (Bochner)}}$ Assume $\underline{Ric \leq 0 \text{ on cpt. oriented}}$.

Then $\underline{\text{every killing vector field is parallel.}}$ $\Rightarrow$ no continuous family of isometries.

If $Ric < 0$, then every killing vector field vanishes.

$\boxed{\text{Def}}$ A vector field X is a killing vector field if $\underline{\mathcal{L}_X g = 0}$.

($\Leftrightarrow$ If $\sigma(t) = \exp(tX)$ 1-psg. generated by X, then $\sigma(t)^* g = g$)

i.e. $\sigma(t)$ is an isometry

$\underline{\text{Lem}}$: X is killing $\Leftrightarrow$ $\boxed{\begin{array}{l} \nabla X : TM \to TM \text{ is skew-symmetric} \\ \quad\; V \mapsto \nabla_V X \end{array}}$

$$\underbrace{(\nabla_V X, W)}_{\nabla X(V)} = -(V, \underbrace{\nabla_W X}_{\nabla X(W)})$$

$\underline{\text{Pf}}$: X killing $\Leftrightarrow (\mathcal{L}_X g)(V, W) = 0 \quad \forall V, W \in TM$

$$\begin{aligned}
(\mathcal{L}_X g)(V,W) &= \mathcal{L}_X (g(V,W)) - g(\mathcal{L}_X V, W) - g(V, \mathcal{L}_X W) \\
&= X(V, W) - ([X,V], W) - (V, [X,W]) \\
&= (\cancel{\nabla_X V, W}) + (V, \cancel{\nabla_X W}) - ((\cancel{\nabla_X V} - \nabla_V X, W)) - ((V, \cancel{\nabla_X W} - \nabla_W X)) \\
&= (\nabla_V X, W) + (V, \nabla_W X) = 0. \quad \forall V, W \in TM.
\end{aligned}$$

$$((\nabla X)(V), W) + (V, (\nabla X)(W)) = 0 \Leftrightarrow \nabla X : TM \to TM \text{ is skew-symmetric.}$$

Lemma: X is Killing. Set $f=\frac{1}{2}|X|^2$. Then

$$\boxed{\Delta f = \Delta\left(\tfrac{1}{2}|X|_g^2\right) = |\nabla X|_g^2 - \mathrm{Rc}(X,X)}.$$

$(\nabla X : TM \to TM)$

More generally, $\boxed{\mathrm{Hess} f(V,V) = |\nabla_V X|^2 - R(V,X,X,V).\ \forall V\in TM}$

$$\underbrace{\sum_i \mathrm{Hess} f(e_i,e_i)}_{\Delta f} = \underbrace{\sum_i |\nabla_{e_i}X|^2}_{|\nabla X|^2} - \underbrace{\sum_i R(e_i,X,X,e_i)}_{-\mathrm{Rc}(X,X)}.$$

Pf:
$$\mathrm{Hess} f(V,W) = (\nabla df)(V,W) = (\nabla_V df)(W)$$
$$= V(df(W)) - df(\nabla_V W)$$
$$= \underline{V(\nabla f, W)_g - (\nabla f, \nabla_V W)_g} = \underline{(\nabla_V \nabla f, W)}$$

$$(\nabla f, W) = df(W) = W\left(\tfrac{1}{2}|X|^2\right) = W\left(\tfrac{1}{2}(X,X)\right)_g$$
$$= (\underbrace{\nabla_W X}_{(\nabla X)(W)}, X) = -(\nabla_X X, W) \quad \Rightarrow \underline{\nabla f = -\nabla_X X}$$

$$\underbrace{(\nabla_V X - \nabla_X V, \nabla_V X)}_{([V,X], \nabla_V X)}$$

$$\mathrm{Hess} f(V,V) = (\nabla_V(-\nabla_X X), V)$$
$$= \underbrace{-(\nabla_V\nabla_X X - \nabla_X\nabla_V X - \nabla_{[V,X]}X, V)}_{-(R(V,X)X,V)} \underline{-(\nabla_X\nabla_V X, V) - (\nabla_{[V,X]}X, V)}$$
$$= -(R(V,X)X,V) + |\nabla_V X|^2 + \Big(\underbrace{-(\nabla_X\nabla_V X, V) - (\nabla_X V, \nabla_V X)}_{-X((\nabla_V X, V))}\Big)$$

$(\nabla_X V, \nabla_V X) = 0$

$$-X((\nabla_V X, V)), \quad \underline{(\nabla X(V), V)} = 0 \quad \blacksquare$$

$$\bullet \quad \underbrace{\int_M \Delta\left(\tfrac{1}{2}|X|^2\right)}_{\substack{\text{stokes } \| \\ \partial M = \phi \quad 0}} = \underbrace{\int |\nabla X|^2}_{\substack{\vee\!/ \\ 0}} - \underbrace{\int Ric(X,X)}_{\substack{\vee\!/ \ Ric \le 0 \\ 0}} . \Rightarrow \begin{cases} \nabla X \equiv 0 \\ Ric(X,X) \equiv 0 . \end{cases}$$

$\Rightarrow$ X parallel

if $Ric < 0$, then $X \equiv 0$.

Rmk: $\boxed{\begin{matrix} isom(M,g) \\ \| \\ \{\text{killing vector fields}\} \end{matrix}}$ ═ Lie algebra of $\boxed{Isom(M,g)}$

$$Isom(M,g) = \left\{ \begin{matrix} \sigma: M \to M \text{ diffeomorphism} \\ \sigma^* g = g \end{matrix} \right\}$$

X, Y killing $\Rightarrow$ $[X,Y]$ is killing

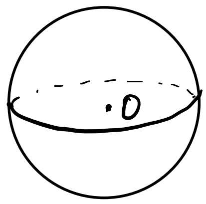
Fact: This is a Lie gp.

If M cpt. then $Isom(g)$ is cpt.

Fact: If M is closed. Then any vector field X generate a 1 psg: $\{\sigma(t),\ t \in \mathbb{R}\}$ $\quad \sigma: (-\infty, +\infty) \longrightarrow Diff(M)$

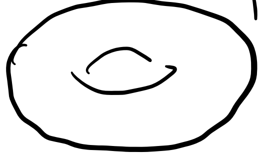
$$\left(\begin{cases} \frac{d}{dt}(\sigma(t)\cdot p) = X(\sigma(t)\cdot p) & \forall p \in M \\ \sigma(0)\cdot p = p \end{cases} \right)$$

Ex:

-  $S^n \subset \mathbb{R}^{n+1}$

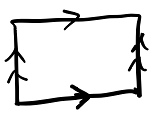
$$\mathrm{Isom}(S^n, g_{round}) = O(n+1)$$

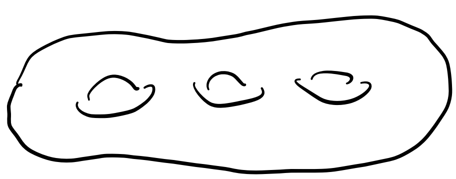
$$\mathrm{isom}(S^n, g_{round}) = so(n+1). \quad \dim = \frac{(n+1)n}{2}.$$

-  T^2

$$\mathrm{Isom}^+(T^2) \cong T^2$$

||



-  $\Rightarrow \mathrm{Isom}(\Sigma_g)$ finite.

Gauss curv. = Rse < 0

<u>Thm</u>: Fix any $p \in M$. Then any Killing field X is determined by $(\underline{X(p)}, \nabla X|_p) \in T_pM \times so(T_pM)$

$\nabla X : T_pM \to T_pM$ skew-symmetric.

$so(T_pM) = \{$skew symmetric linear transf.$\}$.

$\Rightarrow$ An injective map:

$$\begin{aligned} \mathrm{isom}(M,g) &\hookrightarrow T_pM \times so(T_pM) \\ X &\mapsto (X(p), \nabla X|_{T_pM}) \end{aligned}$$

$$\Rightarrow \dim \mathrm{isom}(M,g) \leq n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

"=" holds for (S^n, g_{round})