Thm (Bochner) (M, g) closed Riem. Assume $Ric \geq 0$

Then harmonic 1-forms are parallel. If $Ric > 0$, then all harmonic 1-forms vanish. ($\Rightarrow b_1 = 0$).

$$Ric: T_p^*M \to T_pM, \quad \alpha \mapsto Ric(\alpha), \quad \langle Ric(\alpha), v\rangle_g = Ric(\alpha^\#, v)$$

Pf: Weitzenböck formula: $\boxed{\Delta\omega = \nabla^*\nabla\omega + Ric(\omega)}$.

$$\underbrace{\int(\Delta\omega, \omega)}_{=0} = \underbrace{\int(\nabla^*\nabla\omega, \omega)}_{=\int(\nabla\omega, \nabla\omega)\,\geq\, 0} + \underbrace{\int Ric(\omega^\#, \omega^\#)}_{\geq 0} \Rightarrow \begin{cases} \nabla\omega = 0 \\ Ric(\omega^\#, \omega^\#) = 0. \end{cases} \blacksquare$$

Thm: M closed Riem. mfd. Assume $\mathcal{R}$ the curvature operator is nonnegative. Then all harmonic p-form is parallel. If $\mathcal{R}$ is positive, the harmonic p-forms vanish when $p = 1, \cdots, n-1$. $\Leftrightarrow \mathcal{H}^p = 0,\ p = 1, \cdots, n-1$

$\Updownarrow$

(Rm: $\mathcal{R} > 0 \Rightarrow M \cong S^n$ $\left(\begin{matrix}\text{Böhm} \\ \text{Wilking}\end{matrix}\right)$ $\longrightarrow$ $b_p = 0,\ p = 1, \cdots, n-1.$)

- Weitzenböck formula for p-forms: $\forall\, \omega \in \Omega^p$,

$$\boxed{\Delta\omega = \nabla^*\nabla\omega + F(\omega)}\ ?$$

Vanishing Thm

$$\Delta = dd^* + d^*d = (d + d^*)\circ(d + d^*) = D \circ D$$
$$= d\cdot\delta + \delta\cdot d \quad (d^2 = 0,\ d^{*2} = 0)$$

$$D = d + d^*: \Omega^{\substack{even \\ odd}} \to \Omega^{\substack{odd \\ even}}$$

$$\Omega^\bullet \to \Omega^\bullet$$
$$\Omega^{even} \oplus \Omega^{odd} \qquad \Omega^{odd} \oplus \Omega^{even}$$

$$\boxed{\begin{aligned} \nabla: \Omega^p &\to \Omega^1 \otimes \Omega^p \\ \omega &\mapsto \nabla\omega = \sum_i \theta^i \otimes \nabla_{e_i}\omega \end{aligned}}$$

$\{e_i\}$ o.n. frames for TM $\quad (e_i, e_j)_g = \delta_{ij}$

$\{\theta^i\}$ dual o.n. frame for T^*M $\quad \langle \theta^i, e_j \rangle = \delta_{ij}$

$$\nabla\nabla : \Omega^p \longrightarrow \Omega^1 \otimes \Omega^1 \otimes \Omega^p$$

$$\omega \longmapsto \nabla(\nabla\omega) = \nabla^2\omega$$

$$\boxed{(\nabla_{e_i}\nabla\omega)(v) = \nabla_{e_i}(\nabla\omega(v)) - (\nabla\omega)(\nabla_{e_i}v) = \nabla_{e_i}\nabla_v\omega - \nabla_{\nabla_{e_i}v}\omega.}$$

$$\nabla(\nabla\omega) = \theta^i \otimes \nabla_{e_i}(\nabla\omega) = \theta^i \otimes \theta^j \otimes (\nabla_{e_i}\nabla\omega)(e_j) = \theta^i \otimes \theta^j (\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{\nabla_{e_i}e_j}\omega).$$

$$\nabla^2\omega = \nabla\nabla\omega = \sum_{i,j} \theta^i \otimes \theta^j \otimes \underbrace{(\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{\nabla_{e_i}e_j}\omega)}_{= \nabla^2_{e_i,e_j}\omega}$$

$$\mathrm{tr}(\nabla^2\omega) = g_{ij}\nabla^2_{e_i,e_j}\omega \underset{\{\theta_i\}\ \text{o.n.}}{=} \nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega$$

$$\boxed{\nabla^*\nabla\omega = -\mathrm{tr}(\nabla^2\omega) = -\sum_i (\nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega)}.$$

$$(\nabla^*\nabla\omega, \eta)_{L^2} = -\int_M ((\nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega), \eta)\, d\mathrm{vol}.$$

$$= -\int \left(e_i(\nabla_{e_i}\omega, \eta) - (\nabla_{e_i}\omega, \nabla_{e_i}\eta) - (\nabla_{\nabla_{e_i}e_i}\omega, \eta) \right) d\mathrm{vol}$$

$$= (\nabla\omega, \nabla\eta)_{L^2}$$

$$\left(\alpha = (\nabla\omega, \eta).\ \int_M \mathrm{div}\,\alpha = 0. \right)$$

Lemma: $\boxed{d\omega = \sum_i \theta^i \wedge \nabla_{e_i}\omega}$ $\forall$ p-form ω

$$\boxed{d^*\omega = -\sum_i \iota_{e_i}\nabla_{e_i}\omega}$$

Pf: p-form $\omega = \varphi_1 \wedge \cdots \wedge \varphi_p$ $\quad d\omega = \sum (-1)^{i-1}\varphi_1 \wedge \cdots \wedge d\varphi_i \wedge \cdots \wedge \varphi_p$

$$\nabla_{e_i}\omega = \sum \varphi_1 \wedge \cdots \wedge \nabla_{e_i}\varphi_i \wedge \cdots \wedge \varphi_p$$

$$\theta^i \wedge \nabla_{e_i}\omega = \sum (-1)^{i-1}\varphi_1 \wedge \cdots \wedge \theta^i \wedge \nabla_{e_i}\varphi_i \wedge \cdots \wedge \varphi_p$$

φ 1-form

$$(d\varphi)(e_i, e_j) = e_i(\varphi(e_j)) - e_j(\varphi(e_i)) - \varphi([e_i, e_j])$$

$\nabla = \nabla^{LC}$ is torsion free

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$$

$$= \big(e_i(\varphi(e_j)) - \varphi(\nabla_{e_i} e_j)\big) - \big(e_j(\varphi(e_i)) - \varphi(\nabla_{e_j} e_i)\big)$$

$$= (\nabla_{e_i}\varphi)(e_j) - (\nabla_{e_j}\varphi)(e_i).$$

$$(\theta^k \wedge \nabla_{e_k}\varphi)(e_i, e_j) = \theta^k(e_i)(\nabla_{e_k}\varphi)(e_j) - \theta^k(e_j)\nabla_{e_k}\varphi(e_i)$$

$$= (\nabla_{e_i}\varphi)(e_j) - \nabla_{e_j}\varphi(e_i)$$

$$\Delta = (d+d^*)^2 = D^2, \quad \underline{D = d + d^*} = \Big(\sum_i \theta^i \wedge - \iota_{e_i}\Big) \circ \nabla_{e_i}$$

$$D\omega = \sum_i (\theta^i \wedge - \iota_{e_i}) \nabla_{e_i}\omega = \sum_i \theta^i \bullet \nabla_{e_i}\omega$$

$\varphi \in \Omega^1(M)$. $\varphi\bullet : \Omega^{odd/even} \to \Omega^{even/odd}$. Clifford multiplication

$$\boxed{(\varphi^{\#}, v)_g = \varphi(v)} \qquad \varphi^{\#} \in TM$$

$$\underset{\in \Omega^p}{\omega} \longmapsto \underbrace{\varphi \wedge \omega}_{\Omega^{p+1}} - \underbrace{\iota_{\varphi^{\#}}\omega}_{\Omega^{p-1}} = \varphi \bullet \omega$$

$$(\iota_v \omega)(v_1, \dots, v_{p-1}) = \omega(v, v_1, \dots, v_{p-1}).$$

$\bullet\varphi : \Omega^{odd/even} \to \Omega^{even/odd}$ right Clifford multiplication.

$$\underset{\in \Omega^p}{\omega} \longmapsto (-1)^p(\varphi \wedge \omega + \iota_{\varphi^{\#}}\omega)$$

$$= \omega \wedge \varphi + (-1)^p \iota_{\varphi^{\#}}\omega$$

- $$\boxed{\varphi_1 \bullet \varphi_2 + \varphi_2 \bullet \varphi_1 = -2(\varphi_1, \varphi_2)_g = -2(\varphi_1^{\#}, \varphi_2^{\#})_g}$$

$$\varphi_1 \wedge \varphi_2 + \varphi_2 \wedge \varphi_1 = 0 .$$

$$\begin{cases} \varphi_1 \perp \varphi_2 \Rightarrow \underline{\varphi_1 \bullet \varphi_2 = -\varphi_2 \bullet \varphi_1} \\ \varphi_1 = \varphi_2 = \varphi \Rightarrow \varphi \bullet \varphi = -\underline{(\varphi, \varphi)_g} \end{cases}$$

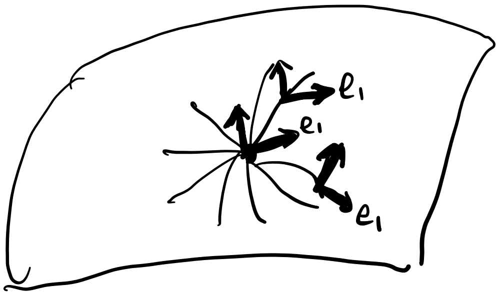
$D\omega = \sum_i \theta^i \bullet \nabla_{e_i} \omega$, Fix any point $p \in M$

$\|$

$(d+d^*)\omega$

Choose $\{e^i\}$ o.n. frame s.t.

$\underline{(\nabla e^i)_p = 0 , \quad i = 1, \cdots, n .}$



$$\underline{\Delta\omega} = D \circ D\omega = D\Big(\sum_j \theta^j \bullet \nabla_{e_j}\omega\Big) = \sum_{i,j} \underline{\theta^i \bullet \nabla_{e_i}\big(\theta^j \bullet \nabla_{e_j}\omega\big)}$$

$\|$

$$\theta^i \bullet \big((\nabla_{e_i}\theta^j) \bullet \nabla_{e_j}\omega\big) + \theta^j \bullet \nabla_{e_i}\nabla_{e_j}\omega\big)$$

$$= \sum_{i,j} \theta^i \bullet \theta^j \bullet \nabla_{e_i}\nabla_{e_j}\omega$$

$$= \sum_{i=j} \theta^i \bullet \theta^j \, \nabla_{e_i}\nabla_{e_j}\omega + \sum_{i<j} \theta^i \bullet \theta^j \bullet \nabla_{e_i}\nabla_{e_j}\omega + \sum_{i>j} \underset{\substack{\| \\ -\theta^j \bullet \theta^i}}{\theta^i \bullet \theta^j} \bullet \nabla_{e_i}\nabla_{e_j}\omega$$

$$= -\nabla_{e_i}\nabla_{e_i}\omega + \sum_{i<j} \theta^i \bullet \theta^j \bullet \underline{\big(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\big)} \quad \nabla_{\nabla_{e_i}e_j - \nabla_{e_j}e_i}$$

$$= \underline{\nabla^*\nabla\omega + \sum_{i<j} \theta^i \bullet \theta^j \bullet R(e_i, e_j)\,\omega}$$

Similarly $\Delta w = \nabla^*\nabla w + \sum_{i<j} R(e_i, e_j) w \bullet \theta^j \bullet \theta^i$

$$\Delta w = \nabla^*\nabla w - \sum_{i<j} R(e_i, e_j) w \bullet \theta^i \bullet \theta^j$$

$$\boxed{\Delta w = \nabla^*\nabla w + \sum_{i<j} \theta^i \bullet \theta^j \bullet R(e_i, e_j) w}$$

$$\Delta w = \nabla^*\nabla w + \frac{1}{2}\sum_{i<j} \underbrace{[\theta^i \bullet \theta^j, R(e_i, e_j) w]}_{\theta^i \bullet \theta^j \bullet R(e_i,e_j)w - R(e_i,e_j)w \bullet \theta^i \bullet \theta^j}$$

$$= \nabla^*\nabla w + \frac{1}{4}\sum_{i \neq j} [\theta^i \bullet \theta^j, R(e_i, e_j) w]$$

Ex: w 1-form. $\boxed{\Delta^w = \nabla^*\nabla w + \frac{1}{2} \sum_{i\neq j} \theta^i \bullet \theta^j \bullet R(e_i, e_j) w}$.

$$\frac{1}{2} \theta^i \bullet \theta^j \bullet R(e_i, e_j) w = \sum_{\substack{i\neq j \\ k}} \theta^i \bullet \theta^j \bullet \theta^k (R(e_i,e_j)w, e_k)$$

$$= \frac{1}{2} \sum_{\substack{i\neq j \\ \neq k}} \underbrace{\theta^i \bullet \theta^j \bullet \theta^k}_{\theta^j \bullet \theta^k \bullet \theta^i = \theta^k \bullet \theta^i \bullet \theta^j} (R(e_i,e_j)w, e_k) = \frac{1}{3} \sum_{\substack{i\neq j \\ \neq k}} \theta^i \bullet \theta^j \bullet \theta^k \begin{pmatrix} R(e_i,e_j)w, e_k \\ + \\ R(e_j,e_k)w, e_i \\ + \\ R(e_k,e_i)w, e_j \end{pmatrix} = 0$$

$$+ \frac{1}{2} \sum_{\substack{k=i \\ \neq j}} \underbrace{\theta^i \bullet \theta^j \bullet \theta^i}_{+\theta^j} \cdot \underbrace{(R(e_i,e_j)w, e_i)}_{Ric(e_j, w)}$$

$$+ \frac{1}{2} \sum_{\substack{k=j \\ \neq i}} \theta^i \bullet \underbrace{\theta^j \bullet \theta^j}_{-1} \underbrace{(R(e_i,e_j)w, e_j)}_{-Ric(w^\#, e_i)} = \underbrace{\sum_i \theta^i \cdot Ric(w^\#, e_i)}_{\boxed{Ric(w^\#)}}$$

$$\Delta \omega = \nabla^* \nabla \omega + \frac{1}{4} \sum_{i \neq j} [\theta^i \cdot \theta^j, R(e_i, e_j)\omega]$$

$$(\Delta \omega, \omega)_g = (\nabla^* \nabla \omega, \omega)_g + \frac{1}{4}\left([\theta^i \cdot \theta^j, R(e_i, e_j)\omega], \omega\right)_g$$

$$\| $$

$$\left(ad_{\theta^i \cdot \theta^j}(R(e_i,e_j)\omega), \omega\right)$$

$$\|$$

$$-\left(R(e_i,e_j)\omega, ad_{\theta^i \cdot \theta^j}(\omega)\right)$$

$$\|$$

$$(R(e_i,e_j)e_k, e_l)\, ad_{\theta^k \cdot \theta^l}(\omega)$$

$$\int (\Delta \omega, \omega) = \int (\nabla^* \nabla \omega, \omega) + \frac{1}{4} \int \left((R(e_i,e_j)e_k, e_l)\, ad_{\theta^k \cdot \theta^l}(\omega), ad_{\theta^i \cdot \theta^j}(\omega) \right)$$

$\| \; \Delta\omega = 0$ $\qquad \|$ $\qquad \|$

$$0 \qquad \int (\nabla \omega, \nabla \omega) \qquad (R(e_i \wedge e_j), e_k \wedge e_l)_g$$

$$R : \Lambda^2 TM \times \Lambda^2 TM \to \mathbb{R} \quad \text{symmetric bilinear form}$$

$$\| \qquad (X \wedge Y, Z \wedge W) \mapsto R(X \wedge Y, Z \wedge W) = (R(X,Y)Z, W)_g$$

$$R_g \qquad\qquad \| $$

$$R(Z \wedge W, X \wedge Y)$$

$$\dim = \binom{n}{2} = \frac{n(n-1)}{2}$$

$$R : \Lambda^2 TM \to \Lambda^2 TM \quad \text{self-adjoint linear transformation}$$

$\rightsquigarrow$ R is diagonalizable $\rightsquigarrow$ eigenvalues $\lambda_1 \le \cdots \le \lambda_{\binom{n}{2}}$, $\binom{n}{2} = d$

$$\Theta_1 \quad \cdots \quad \Theta_d$$

<u>Def</u>: R is positive if $\lambda_1 > 0$

nonnegative $\lambda_1 \geq 0$

Curvature Term:

$$\Big(\big(R(e_i \wedge e_j), e_k \wedge e_l \big)_g \, \mathrm{ad}_{\theta^i \cdot \theta^j} \omega , \ \mathrm{ad}_{\theta^k \cdot \theta^l} \omega \Big)_g$$

$$= \Big(\underbrace{\big(\underbrace{R(\Theta_\alpha)}_{= \lambda_\alpha \Theta_\alpha}, \Theta_\beta \big)}_{= \lambda_\alpha \cdot \delta_{\alpha\beta}} \, [\Theta_\alpha, \omega] , \ [\Theta_\beta, \omega] \Big)_g$$

$$= \sum_\alpha \lambda_\alpha \big\| [\Theta_\alpha, \omega] \big\|^2 \geq 0 .$$