(M,g) : closed Riem. mfd. $\Omega^k = \{\text{smooth } k\text{-forms on } M\}$

$$H^k_{dR}(M) = \frac{\mathrm{Ker}(d: \Omega^k \to \Omega^{k+1})}{\mathrm{Im}(d: \Omega^{k-1} \to \Omega^k)} = \frac{\{\omega \text{ k-form} : d\omega = 0\}}{\{d\eta : \eta \in \Omega^{k-1}\}}$$

Thm (De Rham) $H^k_{dR}(M) \cong H^k_{sing}(M) \cong \check{H}^k(M, \mathbb{R}) = H^k(M)$

<u>Def</u>: $\omega \in \Omega^k$ is a harmonic form if $\Delta \omega = 0$ where

$\Delta = d\delta + \delta d$ is Hodge Laplacian operator.

$\underline{\delta = d^*}$ w.r.t. the L^2-inner product induced by the Riem. metric g

$\quad\ \, \shortparallel$

$\pm * d * : \Omega^k \to \Omega^{k-1}$ $\quad (\omega, d\eta)_{L^2} = (d^*\omega, \eta)_{L^2}$

$$\boxed{\Delta \omega = 0 \iff d\omega = 0 \text{ and } \delta\omega = 0.}$$

$$\begin{array}{ccc} \mathcal{H}^k = \{\text{harmonic } k\text{-forms}\} & \longrightarrow & H^k \\ \cup\!\shortmid & & \cup\!\shortmid \\ \omega & \longmapsto & [\omega] \end{array}$$

1935

Thm (Hodge Thm) <u>The above map is an isomorphism.</u>

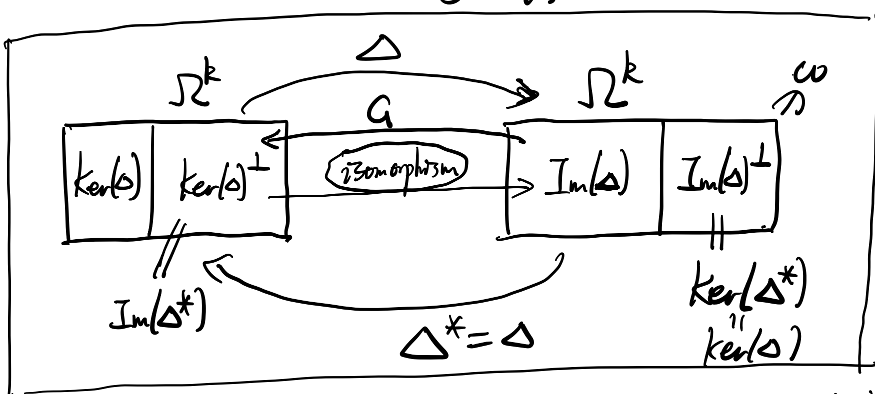
Idea from finite dimensional situation:

$$\Delta : \Omega^k \to \Omega^k. \qquad \Delta^* = \underset{\substack{\shortparallel \\ (dd^*+d^*d)^*}}{(d\delta + \delta d)^*} = \overset{\substack{dd^* + d^*d \\ \shortparallel}}{\delta^* d^* + d^* \delta^*} = \Delta$$

$$\rightsquigarrow \quad \Omega^k = \mathrm{Ker}(\Delta) \oplus \underset{\substack{\shortparallel \\ \underline{\mathrm{Im}(\Delta^*)}}}{\underline{\mathrm{Ker}(\Delta)^\perp}} = \mathrm{Ker}(\Delta) \oplus \underset{\substack{\shortparallel \\ \mathrm{Im}(\Delta)}}{\underline{\mathrm{Im}(\Delta^*)}}$$

- $\underline{\omega \in \ker(\Delta)}$, $\underset{\substack{\| \\ \Delta^* \eta}}{\omega'} \in \operatorname{Im}(\Delta^*) \rightsquigarrow (\omega, \Delta^* \eta) = (\Delta \omega, \eta) = 0$

- $\forall\, \omega \in \Omega^k$, $\omega_1 = \operatorname{Proj}_{\ker(\Delta)} \omega \in \ker(\Delta)$, $\omega - \omega_1 = \Delta^* \eta$

- 

$$\begin{array}{ccc} \omega = & \omega_1 & + \ \omega_2 \\ & G\downarrow & \downarrow \\ \Omega^k \ni & G\omega_1 & + \ 0 \end{array}$$

$$\Omega^k = \ker(\Delta) \oplus \underset{\substack{\| \\ \operatorname{Im}(d\delta + \delta d)}}{\operatorname{Im}(\Delta)}, \qquad \operatorname{Im}(\Delta) = \underline{\operatorname{Im}(d) \oplus \operatorname{Im}(\overset{d^*}{\overset{\|}{\delta}})}$$

$$\Delta \eta = d \delta \eta + \delta d \eta$$

$$(d\eta, \delta\omega) = (dd\eta, \omega) = 0.$$

(*)
$$\boxed{\begin{aligned} \Omega^k &= \underline{\ker(\Delta) \oplus \operatorname{Im} d} \oplus \underline{\operatorname{Im}(\delta)} \\ &= \quad \ker(d) \qquad \oplus \operatorname{Im}(d^*) \end{aligned}}$$

$$\boxed{\mathcal{H}^k = \ker(\Delta) = \frac{\ker(d)}{\operatorname{Im}(d)} = H^k_{dR}}$$

$$T : V \to V$$

$$V = \ker(T) \oplus \underset{\substack{\| \\ \ker(T)^\perp}}{\operatorname{Im}(T^*)}$$

(pseudo-inverse)
(Parametrix)

<u>Thm (Hodge)</u> $\exists$ a Green operator $\underline{G : \Omega^k \to \Omega^k}$ s.t.

$$\underline{G \circ \Delta = \Delta \circ G = Id_{\Omega^k} - \operatorname{Proj}_{\ker(\Delta)}}$$

$\leftarrow$ orthogonal Proj.

- $\Omega^k \rightsquigarrow W^s_k$ = Sobolev space = Completion of Ω^k w.r.t. Sobolev norm

$\cdots \subset W^2_k \subset W^1_k \subset W^0_k = L^2$ space of k-forms

$W^s_k = \{\omega :$ k-form, L^2-integrable, $\underbrace{\nabla \cdots \nabla}_{s\text{-times}} \omega \in L^2$ in the sence of currents $\}$

$$\Omega^k = \bigcap_{s \geq 0} W_k^s$$

Ref.: Jost: Geometric Analysis

- $$\boxed{\Delta : W_k^s \longrightarrow W_k^{s-2}}$$
$$\begin{array}{ccc} \cup & & \cup \\ \Omega^k & \xrightarrow{\Delta} & \Omega^k \end{array}$$

$G : W_k^{s-2} \longrightarrow W_k^s$ $\boxed{\text{Smoothing operator}}$

$G\Delta = \Delta G = \mathrm{Id} - \mathrm{Proj.}\ker\Delta$ ⇝ Fredholm operator ⇝ Eigenspaces are finite dim.

- $(\Delta\omega, \omega) = ((dd^* + d^*d)\omega, \omega) = (d^*\omega, d^*\omega) + (d\omega, d\omega) \geq 0$.

Elliptic operator ⇝ $\mathrm{Spec}(\Delta) = \{ \underset{\lambda_1}{\overset{\shortparallel}{0}} = \cdots = \underset{\lambda_p}{\overset{\shortparallel}{0}} < \lambda_{p+1} \leq \cdots\cdots \} \subset \mathbb{R}_{\geq 0}$ discrete.

$E_\lambda = \{\omega : \Delta\omega = \lambda\omega\}$ is finite dimensional.

$\omega = \sum_i f_i \omega_i \quad \{\omega_i \in E_{\lambda_i} : \text{o.n.b.}\}$.

$$G\omega = \sum_{i>p} \frac{1}{\lambda_i}\omega_i = \sum_{i>p} \underset{f_i}{\underset{\shortparallel}{(\omega, \omega_i)}} \frac{1}{\lambda_i}\omega_i$$

$$= \int_X \Big(\sum_{i>p} \omega(y)\, \omega_i(y) \cdot \frac{1}{\lambda_i}\Big) \omega_i(x)\, dy$$

$$\boxed{\int_X \omega(y) \cdot K(x,y)\, dy}$$

Regularity Lemma: If $\Delta\omega \in \underset{\text{smooth}}{\Omega^k}$, then $\omega \in \underset{\text{smooth}}{\Omega^k}$.

In particular, $\ker(\Delta)$ consists of smooth harmonic forms.

- Bochner Technique

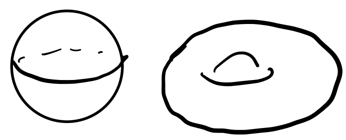
<u>Thm (Bochner)</u> 1. (M,g) closed Riem. mfd. Assume $\underline{\underline{Ric(g) \geq 0}}$

Then <u>every harmonic 1-form is parallel.</u>

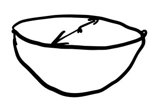
$$\Downarrow$$

$b_1(M) = \dim H^1(M) \leq n$. <u>"=" holds iff M is a torus.</u>

<u>$n=2$</u>: $b_1(M) \leq 2$.



$b_1 = 0$ $\quad$ $b_1 = 2$



$\mathbb{R}P^2$

Klein bottle

$\nexists$ $Ric \geq 0$

$b_1 = 2g = 2 \cdot$ genus

. . .

2. <u>$Ric(g) \geq 0$ and strictly positive at one point</u>, then all harmonic 1-form vanish. $\left(\xRightarrow{\text{Hodge}} \; \begin{array}{c} H^1(M) = 0 \\ \text{\scriptsize 1/2} \\ \mathcal{H}^1(M) \end{array} \right)$

(<u>Rmk</u>: $\dim = 3$. Hamilton: $Ric > 0 \Rightarrow \tilde{M}$ diffeomorphic to S^3.
(Ricci flow))

- $\underset{\parallel}{\overset{d^*d + dd^*}{}}$ $\Delta : \Omega^k \to \Omega^k$ $\quad$ <u>Hodge Laplacian</u> $\rightsquigarrow$ $\{\overset{\ker}{\underset{\parallel}{\text{harmonic forms}}}\}$

- $\Omega^k \xrightarrow{\nabla = \nabla^{LC}} \Omega^1 \otimes \Omega^k \xrightarrow{\nabla} \Omega^1 \otimes \Omega^1 \otimes \Omega^k \xrightarrow{tr} \Omega^k$

$\nabla^* \nabla = tr(\nabla \nabla \cdot)$ $\quad$ connection Laplacian

$\ker(\nabla^*\nabla) = \{ \text{parallel } k\text{-forms} \} = \{\omega : \nabla\omega = 0\}$

$\rotatebox{90}{$\in$}$
ω $\quad 0 = \underline{(\nabla^*\nabla\omega, \omega)}_{L^2} = \underline{[\nabla\omega, \nabla\omega]}_{L^2} \Rightarrow \nabla\omega \equiv 0$

- Weitzenböck formula for 1-form:

$$\boxed{\Delta = \nabla^*\nabla + Ric}$$

$Ric(v) = \sum_i \langle R(v, e_i)e_i, v\rangle$. $\quad Ric(v_1, v_2) = \sum_i \langle R(v_1, e_i)e_j, v_2\rangle$

$Ric : \underline{T_pM \times T_pM} \to \mathbb{R}$ symmetric bilinear form

$\frac{\partial}{\partial x_i}$ $\{v_i, i = 1, \cdots, n\}$ basis of $T_pM \rightsquigarrow (Ric(v_i, v_j)) = (R_{ij})$ symmetric (n×n)-matrix.

dx_i $\{v_i^*, i = 1, \cdots, n\}$ dual basis of T_p^*M.

$Ric : T_p^*M \to T_p^*M$ $\quad . \quad \omega = \sum_i f_i v_i^*$

$Ric(\omega) = (R_{ij}\, g^{jk} f_k) \cdot v_i^* \overset{\text{o.n.b.}}{=\!=\!=} \sum_{ij} (R_{ij} f_j) \cdot v_i^*$

$Ric(\omega)(v) = Ric(\omega^b, v)$

- $b : T_p^*M \to T_pM$ $\qquad \omega(v) = \langle \omega^b, v\rangle_g$ ← inner product.
- $\# : T_pM \to T_p^*M$ $\qquad \langle v_1, v_2\rangle_g = v_1^\#(v_2)$

Pf of Bochner: $\underline{\Delta\omega = 0}$. $\Delta\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega)$

$$\int(\Delta\omega, \omega)\,dvol = \int(\nabla^*\nabla\omega, \omega)\,dvol + \int\underline{(\mathrm{Ric}(\omega), \omega)}\,dvol$$

$$\begin{array}{ccc} \| & \| & \| \\ 0 & \int(\nabla\omega, \nabla\omega)\,dvol & \int\underline{\mathrm{Ric}(\omega^\flat, \omega^\flat)}\,dvol. \\ & \| & \vee\!/ \\ & \underline{\int\|\nabla\omega\|^2 dvol} & \underline{0} \end{array}$$

$\Longrightarrow$ $\|\nabla\omega\| \equiv 0$, $\underline{\mathrm{Ric}(\omega^\flat, \omega^\flat) \equiv 0}$.

$\Updownarrow$

$\underline{\nabla\omega = 0 \text{ i.e. } \omega \text{ is parallel.}}$

$\underset{\Longrightarrow}{\mathrm{Ric}_p > 0}$ $\underline{\omega^\flat(p) = 0}$ $\xrightarrow{\omega \text{ parallel}}$ $\omega^\flat \equiv 0 \Leftrightarrow \omega = 0$.