Goal: Understand the relationship between curvature and topology.

How curvature conditions put constraint on the topology.

- **Sphere Thm:** M closed Riem. mfd. simply connected $1 < \sec \le 4 \Rightarrow M$ is homeomorphic to S^n

  $\pi_k(M,p) = \{e\}\ \forall 1 \le k \le n-1$

- **Thm (Synge)** If $\sec > 0$, then

  (1) If $\dim M$ is even and orientable, then M is simply connected ($\pi_1(M) =$ trivial)

  (2) If $\dim M$ is odd, then M is orientable.

Hodge Theory: $\boxed{\text{represent cohomology by harmonic forms}}$

(Algebraic Topology: singular cohomology $H^k(M,\mathbb{R}) = \dfrac{\{\text{closed } k\text{-cochain}\}}{\{\text{exact } k\text{-cochain}\}}$)

De Rham Cohomology: M, TM, $T^*M \leftarrow$ cotangent bundle

$\Lambda^k T^*M$: alternating k-forms on M.

$\Omega^k =$ smooth k-forms on $M = \{\text{smooth sections of } \Lambda^k T^*M\}$

$$\omega : \overbrace{T_pM \times \cdots \times T_pM}^{k} \longrightarrow \mathbb{R}$$

$$(V_1, \cdots, V_k) \mapsto \omega(V_1, \cdots, V_k)$$

$$\omega(\cdots, V_i, \cdots, V_j, \cdots) = -\omega(\cdots, V_j, \cdots, V_i, \cdots)$$

$k=0$: $\Omega^0 = \{\text{smooth functions}\}$

$$\wedge : \Lambda^k T^*M \times \Lambda^l T^*M \to \Lambda^{k+l} T^*M$$

$$(\omega \quad , \quad \eta) \mapsto \omega \wedge \eta = \mathcal{A}(\omega \otimes \eta)$$

$$\omega \wedge \eta = (-1)^{k \cdot l} \eta \wedge \omega$$

basis for $\Lambda^1 T^*M$: $dx_1, \cdots, dx_n$

for $\underline{\Lambda^k T^*M}$: $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \ 1 \le i_1 < \cdots < i_k \le n\}$

$$\dim = \binom{n}{k}$$

Locally, $\{U, \{x_i\}\}$ local coordinates, $\omega = \underline{\sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}}$

smooth fct. in $x = (x_1, \cdots, x_n)$

- $d : \Omega^k(M) \to \Omega^{k+1}(M)$ (Exterior differential)

$$\omega \mapsto d\omega = \underline{\sum_{i_1 < \cdots < i_k} d\omega_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}}$$

$$d : \Omega^0(M) \to \Omega^1(M)$$

$$f \mapsto df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$\boxed{d^2 = 0}$$

$$d \cdot \underline{d\eta} = 0$$

$$d(\underset{\Omega^k}{\omega} \wedge \underset{\Omega^l}{\eta}) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta \quad \text{(Leibniz rule)}$$

$\deg \omega = k$

- g: Riemannian metric = inner product on T_pM, $\forall p\in M$

$\rightsquigarrow$ inner product on $\Lambda^k T_p^*M$ $\forall k\geq 1$, $\forall p \in M$

o.n.b. $\{v_1, \cdots, v_n\}$ for T_pM $\rightsquigarrow$ dual basis $\{\sigma_1, \cdots, \sigma_n\}$ for T_p^*M

Set $\{\sigma_{i_1}\wedge\cdots\wedge\sigma_{i_k},\ 1\leq i_1<\cdots<i_k\leq n\}$ to be an o.n.b.

for $\Lambda^k T_p^*M$ $\rightsquigarrow$ well defined inner product on $\Lambda^k T_p^*M$.

- Hodge $*$-operator:

$$\boxed{\begin{aligned} *: \Lambda^k T_p^*M &\longrightarrow \Lambda^{n-k}T_p^*M \\ \sigma_{i_1}\wedge\cdots\wedge\sigma_{i_k} &\longmapsto (-1)^{\mathrm{sgn}(I\cdot J)}\,\sigma_J \\ \| \qquad\qquad & \qquad\qquad \| \\ \sigma_I \qquad\qquad & \qquad \sigma_{j_1}\wedge\cdots\wedge\sigma_{j_{n-k}} \end{aligned}}$$

$\{1,\cdots,n\}\supset I=\{i_1,\cdots,i_k\}$ ordered

$I^c=\{j_1,\cdots,j_{n-k}\}$ ordered

$n=3$: $*\sigma_1=\sigma_{23}=\sigma_2\wedge\sigma_3$. $*\sigma_2=-\sigma_1\wedge\sigma_3=-\sigma_{13}$

$*\sigma_3=\sigma_1\wedge\sigma_2=\sigma_{12}$

$$d\mathrm{vol}(g)=\sqrt{\det(g_{ij})}\,dx_1\wedge\cdots\wedge dx_n$$

$g=\sum_{i,j} g_{ij}\,dx_i\otimes dx_j$

n-form volume form $\|$

$\sigma_1\wedge\sigma_2\wedge\cdots\wedge\sigma_n$

$g'=S\cdot g\cdot S^T$

$$* : \underline{\Lambda^k} \longrightarrow \underline{\Lambda^{n-k}}.$$

$$\omega_2 \longmapsto *\omega_2$$

For any k-form ω_1

$$\boxed{\underset{\text{k-form}}{\omega_1} \wedge \underset{\text{(n-k)-form}}{\underline{*\omega_2}} = \underset{\text{inner product}}{(\omega_1, \omega_2)}\, \underline{dvol}.}$$

n-form

$$(*\omega_1, *\omega_2) = (\omega_1, \omega_2)$$

$$\Lambda^k \xrightarrow{*} \Lambda^{n-k} \xrightarrow{*} \Lambda^k. \qquad \boxed{*^2 = (-1)^{k(n-k)}}$$

$$**(\sigma_1 \wedge \cdots \wedge \sigma_k) = *(\underline{\sigma_{k+1} \wedge \cdots \wedge \sigma_n}) = (-1)^{k(n-k)} \underline{\sigma_1 \wedge \cdots \wedge \sigma_k}$$

$$(k+1 \;\; k+2 \; \cdots \; n \;\; 1 \;\; 2 \; \cdots \; k)$$

Inner product on $\underline{\Omega^k(M)}$: Assume M is closed

$$\boxed{(\omega, \eta) = \int_M (\omega, \eta)_p \, dvol = \int_M \underline{\omega \wedge * \eta}}$$

- Adjoint of $d : \Omega^k(M) \to \Omega^{k+1}(M)$

$\delta : \Omega^{k+1}(M) \to \Omega^k(M)$

$$\boxed{(d\omega, \eta) = (\omega, \delta\eta)} \quad \forall \ \omega \in \Omega^k, \ \eta \in \Omega^{k+1}.$$

$$(d\omega, \eta) = \int_M d\omega \wedge *\eta = \int_M \underline{d(\omega \wedge *\eta)} - (-1)^k \omega \wedge d*\eta$$

$$= \cancel{\int_{\partial M} \omega \wedge *\eta} + (-1)^{k+1} \int \underline{\omega} \wedge d*\eta$$

$\partial M = \phi$

$d*\eta = (-1)^{k(n-k)} *(*d*\eta)$

η: $(k+1)\text{-form} \xrightarrow{*} (n-k-1)\text{-form} \xrightarrow{d} (n-k)\text{-form}$

$$= (-1)^{k+1+k(n-k)} \int \underline{(\omega, *d*\eta)} \, dvol$$

$$= (\omega, \delta\eta) = \int \underline{(\omega, \delta\eta)} \, dvol$$

$$\left(\begin{array}{l} \delta = (-1)^{k+1+k(n-k)} \, \underline{*d*} \quad \text{on } (k+1)\text{-form.} \\ \delta = \underline{(-1)^{k+(k-1)(n-k+1)}} *d* \quad \text{on } k\text{-form} \\ k + (k-1)(n-k) + k - 1 = (k-1)(n-k) - 1 = (-1)^{(n-1)n-1} \\ (k-1)(n-k)-1 = (k-1)n - k(k-1) - 1 \end{array} \right)$$

- De Rham Cohomology:

$$\underset{\substack{\| \\ H^k(M)}}{H^k_{dR}(M)} = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}} = \frac{\{\omega \in \Omega^k : d\omega = 0\}}{\{d\eta : \eta \in \Omega^{k-1}\}}$$

exact form.

const. sheaf

Thm (De Rham) $\underline{H^k_{dR}(M)} \cong H^k_{sing}(M) \cong H^k(M, \mathbb{R})$

$\uparrow$ Singular cohomology

$$\| \quad \left(\frac{\{\text{closed } k\text{-cochain}\}}{\{\text{exact } k\text{-cochain}\}} \right)$$

$$\Longrightarrow \dim H^k_{dR}(M) = \dim H^k_{sing}(M) < +\infty.$$

- $\underset{\substack{\uparrow \\ H^k_{dR}}}{[\omega]} = \{ \underset{\substack{| \\ \text{closed}}}{\omega + d\eta} : \forall \eta \in \Omega^{k-1} \}$

$[\omega_2] = [\omega_1] \Longleftrightarrow \omega_2 - \omega_1 = d\eta$

- Q: Is there an $\underline{\text{optimal}}$ representative in $[\omega]$?

$\Updownarrow$

$$\inf_{\eta \in \Omega^k} \|\omega + d\eta\|^2 \overset{?}{=} \omega$$

$$\|\omega + d\eta\|^2 = (\omega + d\eta, \omega + d\eta)$$

$$t \mapsto (\omega + t\cdot d\eta, \omega + t d\eta) \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} (\omega, d\eta) + (d\eta, \omega) = 0$$

obtains minimum $\forall \eta$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; 2(\omega, d\eta)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\boxed{\begin{cases} \delta\omega = 0 \\ d\omega = 0 \end{cases}} \Longleftarrow \quad 0 = 2(\underset{\Omega^{k-1}}{\underset{\Uparrow}{\underline{\delta\omega}}}, \underset{\Omega^{k-1}}{\underset{\Uparrow}{\eta}}), \forall \eta$$

Def: $\Delta : \Omega^k \to \Omega^k$. $\Delta = \underline{d\delta} + \underline{\delta d}$.

Lem: $\Delta\omega = 0 \Longleftrightarrow \underline{d\omega = 0 \text{ and } \delta\omega = 0}$.

Pf: $(\Delta\omega, \omega) = (\underline{d}\delta\omega + \delta d\omega, \omega) = \underset{\|\delta\omega\|^2}{\underline{(\delta\omega, \delta\omega)}} + \underset{\|d\omega\|^2}{\underline{(d\omega, d\omega)}}$

Def: ω is a harmonic form if $\Delta\omega = 0$.

$$\mathcal{H}^k = \{\omega \in \Omega^k(M) : \underset{\underline{d\omega = 0 = \delta\omega}}{\underset{\Updownarrow}{\Delta\omega = 0}}\} \quad \text{space of harmonic forms.}$$

$$\downarrow \qquad \rotatebox{90}{$\supseteq$}$$

$$H^k_{dR}(M) \ni [\omega]$$

Thm (Hodge Thm): If M is closed Riem. mfd, Then $\mathcal{H}^k \to H^k_{dR}$ is an isomorphism.

$$\begin{array}{ccc} \mathcal{H}^k & \longrightarrow & H^k_{dR} \\ \cup\!\!\!| & & \cup\!\!\!| \\ \omega & \longmapsto & [\omega] \end{array}$$

$\uparrow$ harmonic

Rmk:

In particular, inside each $[\omega]$, there is a unique harmonic form.

$\omega_2 - \omega_1 = d\eta$. $\quad (d\eta, d\eta) = (d\eta, \omega_2 - \omega_1)$

ω_1, ω_2 harmonic $\quad\quad = (\eta, \delta\omega_2 - \delta\omega_1) = 0$

$\Rightarrow d\eta = 0 \Rightarrow \omega_2 = \omega_1$

Thm: There is a decomposition:

$$\Omega^k = \mathcal{H}^k \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(\delta)$$

$\mathcal{H}^k \oplus \mathrm{Im}(d)$: d-closed form

$\mathrm{Im}(d)$: d-exact form

$$\mathcal{H}^k = \frac{\mathcal{H}^k \oplus \mathrm{Im}\, d}{\mathrm{Im}\, d}$$