Thm (Rauch-Berger-Klingenberg) M : closed simply connected Riem. mfd. dim n. Assume $1 < \sec \le 4$. Then M is homotopic to a sphere.

quarter pinched $\boxed{1 < \frac{\max \sec}{\min \sec} \le 4}$

$$M \underset{f'}{\overset{f}{\rightleftarrows}} S^n \quad (f \circ f' \sim Id_{S^n},\ f' \circ f \sim Id_M)$$

($\Rightarrow$ M is homeomorphic to S^n by the Topological Poincaré Conj)

(Milnor's exotic sphere) $\not\Rightarrow$ diffeomorphic to S^n

Thm: Smale $n \geq 5$

Freedman $n = 4$

(diffeomorphic) Perelman $n = 3$

✓ $n = 2$.

Ricci flow

Rmk: In fact, M is diffeomorphic to S^n (Brendle-Schoen)

Rmk: $g \rightsquigarrow \lambda \cdot g$, $\sec(g) \rightsquigarrow \lambda^{-1} \sec(g)$

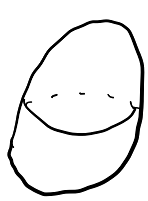
$$\frac{\langle R(X,Y)Y, X \rangle_g}{\|X \wedge Y\|_g^2} \rightsquigarrow \frac{\lambda \cdot \langle\ \rangle}{\lambda^2 \|\cdot\|^2}$$

$$\|X \wedge Y\|_g^2 = \|X\|^2 \|Y\|^2 - \langle X, Y \rangle_g^2$$

$1 \le \frac{\max \sec}{\min \sec} < 4 \leftarrow$ optimal because $(\mathbb{CP}^n, g_{FS})$ ratio $= 4$.

(2n)-dim

at least in even dimension



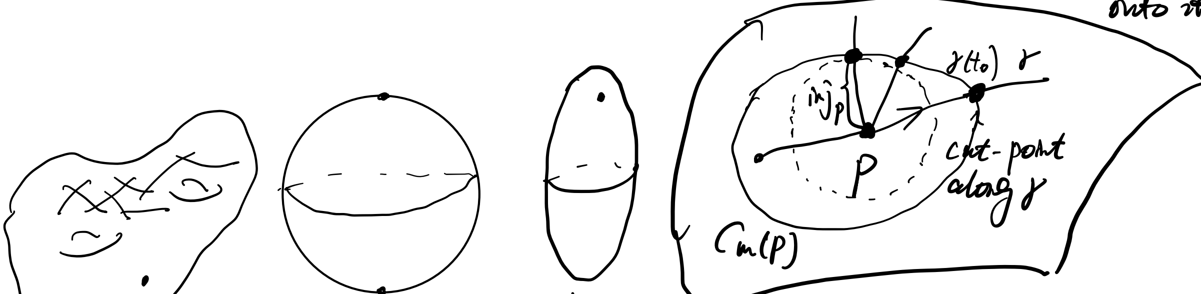
Symmetric space: $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{OP}^2$

(Klingenberg)

Thm: M^n, $n \geq 3$ simply connected, closed Riem. mfd. s.t.

$\underline{1 < \sec \leq 4}$. Then $\boxed{\operatorname{inj}(M) \geq \frac{\pi}{2}}$.

$\underline{\operatorname{inj}(M)} = \inf_{p \in M} \operatorname{inj}_p$, $\operatorname{inj}_p = d(p, \underline{C_m(p)}) = \sup\{r \text{ satisfying } \exp_p : B_r(0) \to M \text{ is } \underline{\text{diffeomorphism}} \text{ onto its image}\}$



cut locus

(Jacobi) $t_0 = \max\{t : \gamma|[0,t] \text{ length minimizing}\}$

$\underline{\operatorname{inj}_p} = d(p, \underline{C_m(p)}) = d(p, q)$. $\Rightarrow$ q satisfies

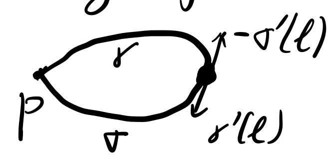
$\uparrow$ closed subset

1, $\exists$ a minimizing geodesic $\gamma : p \to q$ s.t. q is a $\underline{\text{conjugate point along } \gamma}$

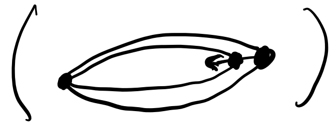
OR

2, $\exists$ exactly 2 minimizing geodesics

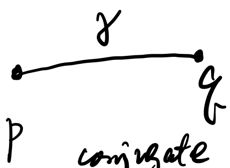
$\boxed{\text{geodesic loop}}$



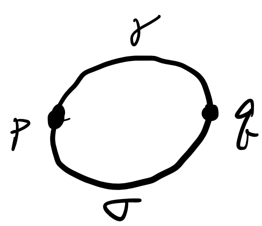
$\operatorname{inj}(M) = \inf_{p \in M} \operatorname{inj}_p = \underline{d(p, q)}$



Either



OR



closed geodesic.

$L(\gamma) = L(\sigma) = l$.

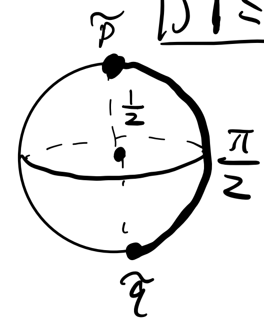
$\Rightarrow \operatorname{inj} = \frac{1}{2} L(\gamma \cup \sigma) = l$.

Prop: $0 < \sec \le 4$. Then

Either $i(M) \ge \frac{\pi}{2}$ OR there is a closed geodesic γ s.t. $i(M) = \frac{1}{2} L(\gamma)$.

$\left(1 < \sec \le 4 \Rightarrow \text{ if } q \text{ is conjugate to } p, \text{ then } d(p,q) \ge \frac{\pi}{2} \right)$

(Rauch comparison Thm: $\widetilde{\sec} \ge \sec \Downarrow |\tilde{J}| \le |J|$)



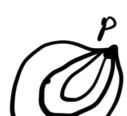
Thm (Klingenberg) The following is true if $n = \dim M$ is even.

$$\boxed{0 < \sec \le 4 \Rightarrow inj(M) \ge \frac{\pi}{2}}$$

- By using some algebraic topology (Hopf, Hurewicz, Whitehead) it is enough to prove $\pi_k(M, p)$ vanishes for $1 \le k \le n-1$

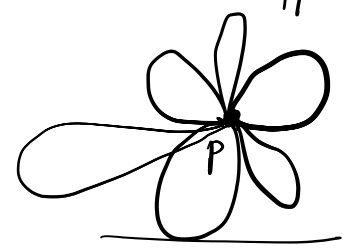
(Results from homotopy Theory)

k-th fundamental gp. $= \{$continuous maps $\alpha : (S^k, 0) \to (M, p)\}/\sim$

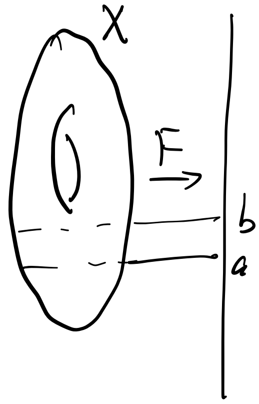
$k=1$: $\pi_1(M,p) = \{$  $\}/\sim$

- $\pi_k(M, p) = \pi_{k-1}(\Omega_{p,p}(M), \{p\}) \neq \{e\}$ trivial gp.

constant loop



- Morse Theory: $F: X \to \mathbb{R}$ a function.



If all critical points in $F^{-1}([a,b])$ have index $\geq m$, Then

$$\pi_k(f^{-1}((-\infty, b]), f^{-1}((-\infty, a])) = 0$$

for any $k \leq m-1$.

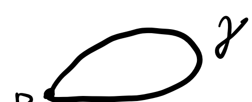
$\pi_k(M, A)$

$$X = \Omega_{p,p} \ni c, \quad F(c) = E(c) = \int_0^1 |c'(t)|^2 dt \geq 0 \quad a = 0.$$

$= \{c : [0,1] \to M,\ c(0) = c(1) = p\}$

$$F^{-1}((-\infty, 0]) = \{p\}.$$

Critical points of E = {geodesic loops based at p}



$$\mathrm{Ind}(\gamma) = \mathrm{Ind}(I_1) = \max\{\dim W : I_1|_W \text{ is negative definite}\}$$

$$I_1(V,V) = \int_0^1 (\langle V', V' \rangle - \langle R(\gamma', V)V, \gamma' \rangle)\, dt$$

It is enough to show $\mathrm{Ind}(\gamma) \geq n-1$.

geodesic loop

(Klingenberg)

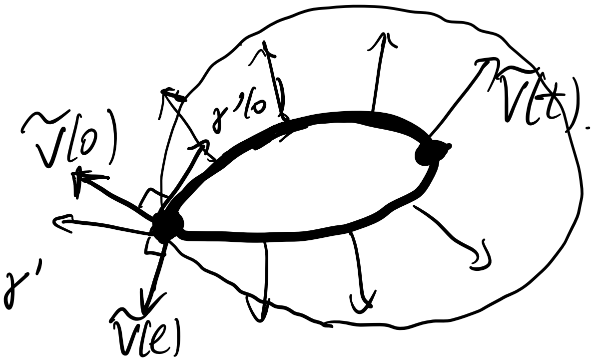
- $\mathrm{Inj}(M) > \frac{\pi}{2} \Rightarrow$ any geodesic loop has length $> \pi$

$\Uparrow$

$(1 \leq) \sec < 4$

$\Updownarrow$

$\left(1 < \sec \leq 4\right)$



- Use $\sec \geq 1$. $\quad V \perp \gamma'$

$$\frac{1}{2}\frac{d^2}{ds^2}E(\gamma_s) = \int_0^1 \left(\langle V', V'\rangle - \langle R(\gamma', V)V, \gamma'\rangle\right) dt$$

$$\langle R(\gamma', V)V, \gamma'\rangle = \sec(\gamma', V)\cdot(|\gamma'|^2 \cdot |V|^2)$$

$$\overset{\sec \geq 1}{\leq} \int_0^1 \left(\langle V', V'\rangle - |\gamma'|^2 \cdot |V|^2\right) dt.$$

$\tilde{V}(t)$: parallel along γ, $\quad V(t) \perp \gamma'(t)$.

$$V(t) = \sin(\pi t)\,\tilde{V}(t). \qquad V(0) = 0 = V(1)$$

$$V' = \cos(\pi t)\,\pi \cdot \tilde{V}(t) + 0, \quad \langle V', V'\rangle = \pi^2 \cos^2(\pi t)|\tilde{V}|^2$$

$$|\gamma'|^2 \cdot |V|^2 = \ell^2 \cdot \sin^2(\pi t) \cdot |\tilde{V}(t)|^2$$

$$\int_0^1 \underline{\left(\pi^2\cos^2(\pi t) - \ell^2 \sin^2(\pi t)\right)} \, \underline{|V(t)|^2} \, dt$$

$$= a \cdot \int_0^1 \left(\pi^2 \cdot \frac{1+\cos(2\pi t)}{2} - \ell^2 \cdot \frac{1-\cos(2\pi t)}{2}\right) dt$$

$$= \frac{a}{2} \cdot (\pi^2 - \ell^2) < 0$$

The 2nd variation is negative definite !

$\ell = L(\gamma)$ length of geodesic loop $> \pi$.

$\Longrightarrow$ $\gamma'(0)^\perp \subset T_pM \rightsquigarrow$ $(n-1)$-dim. subspace of $\mathcal{V}$

for which I_1 is negative definite

$\Longrightarrow$ $\operatorname{Ind}(I_1) \geq n-1$.

$\Longrightarrow$ $\pi_k(\Omega_{p,p}, p) = 0, \quad k \leq (n-1)-1$

$\qquad \| $

$\pi_{k+1}(M, p) = 0, \quad 1 \leq k+1 \leq n-1$

$\Longrightarrow$ M is $(n-1)$-connected

Hopf, Hurewicz, Whitehead

$\Longrightarrow$ M is homotopic to S^n $\xRightarrow{\text{Top. Poincaré Conj.}}$ M is homeomorphic to S^n. ■