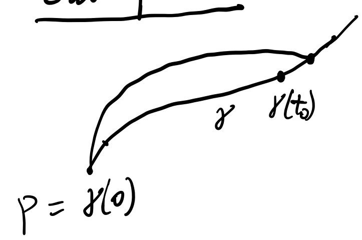
Cut points

M complete



$\gamma(t_0)$ is the cut point of $\gamma(0)$ along γ if

no cut-point

cut value $t_0 = \sup\{t : \gamma|_{[0,t]} \text{ is minimizing}\} \in (0, +\infty]$

M cpt. $\Longleftrightarrow$ Every geodesic starting from p has a cut-point.

("$\Longleftarrow$" non-cpt. $\Rightarrow$ γ has no cut-point.)

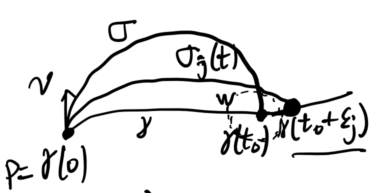
Prop: At cut-point $\gamma(t_0)$, one of two condition hold:

(Klingenberg) 1. $\gamma(t_0)$ is the 1st conjugate point along γ.

2. $\exists$ geodesic $\sigma \not\equiv \gamma$, s.t. $\ell(\sigma) = \ell(\gamma)$.

$\boxed{\text{Conversely if 1 or 2 condition hold, then cut-value} < t_0}$

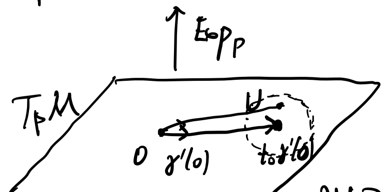
Pf: Assume 1. does not hold, we need to show 2 holds.



$\sigma_j(t) = \exp_p(t \cdot v_j)$ $\quad v_j \in T_pM,\ |v_j| = 1.$

$v_j \neq \gamma'(0)$, $\quad v_j \in S^{n-1}$

As $\varepsilon_j \to 0$, $v_j \to v \in S^{n-1} \subset T_pM$.



$\boxed{\mathrm{Exp}_p : U \xrightarrow{W} M \text{ is a diff}} \Longleftarrow$ $\gamma(t_0)$ is not conjugate pt.

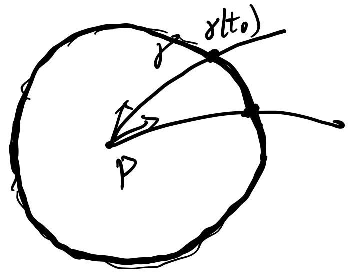
$$W \ni \sigma_j(t_0 + \varepsilon_j') = \gamma(t_0 + \varepsilon_j).$$

$$\boxed{\mathrm{Exp}_p\big((t_0+\varepsilon_j')\underbrace{\sigma_j'(0)}_{v_j}\big)} \qquad \exp_p\big((t_0+\varepsilon_j)\gamma'(0)\big)$$

$\sigma'(0) \neq \gamma'(0)$. If $\sigma'(0)=\gamma'(0)$, then $\boxed{v_j} \to \gamma'(0) \in U$

$$\Rightarrow \quad (t_0+\varepsilon_j')\,\sigma_j'(0) \in U \;\Rightarrow\; \boxed{(t_0+\varepsilon_j')\,\sigma_j'(0) = (t_0+\varepsilon_j)\,\gamma'(0)}$$

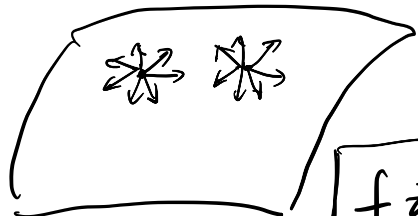
$$\Rightarrow \quad \sigma_j = \gamma .$$ ∎



Fact: Define

$$f : \underset{\{v \in TM,\ |v|=1\}}{\underset{\shortparallel}{T_1M}} \to [0,+\infty]$$

$$f(\gamma(0),\gamma'(0)) = \begin{cases} t_0 & \text{if } \gamma(t_0) \text{ is the cut-pt. of } \gamma(0) \text{ along } \gamma \\ +\infty & \text{if no cut-pt. along } \gamma. \end{cases}$$



$\boxed{f \text{ is then a continuous function on } T_1M}$

Cut locus of P : $C_m(P) = \left\{ \gamma(t_0) : \begin{array}{l} \gamma(t_0) \text{ is the cut point of } \gamma(0)=P \\ \text{along } \gamma \end{array} \right\}$

$$= \{ \gamma(t) : t = f(P, \underset{S^{n-1} \in T_pM}{\underset{\Uparrow}{\gamma'(0)}}) \}$$ is a closed subset of M

$\Rightarrow$ If M is cpt, then $C_m(P)$ is also compact.

$$\mathrm{inj}_P = d(P, C_m(P)) = \min\{ f(P,v) : v \in S^{n-1} \subset T_pM \}$$

$$= \boxed{\underset{\ell}{\underset{\shortparallel}{d(P, q)}} \text{ for some } \underset{\text{closest cut point to } P.}{\underset{\uparrow}{q}} \in C_m(P)} \quad \text{if } C_m(P) \neq \phi.$$

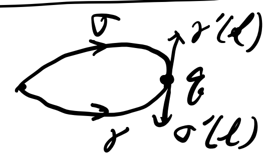
Such a q is a cut point along some geodesic γ.

One of Two conditions hold:

1. Either there exists a minimizing geodesic γ from p to q s.t. q is conjugate to p along γ

2. There exist Exactly 2 minimizing geodesics from p to q. γ and σ

   s.t. $\gamma'(l) = -\sigma'(l)$, $l = d(p,q)$



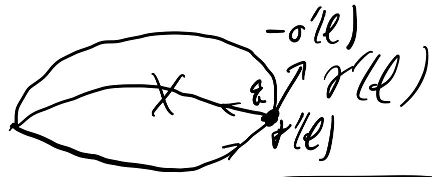
Proof: Let γ be a minimizing geodesic from p to $q = \gamma(t_0)$ and $\gamma(t_0)$ is the cut point.

$\Rightarrow$ Either $\gamma(t_0)$ is conjugate to $\gamma(0)$ OR

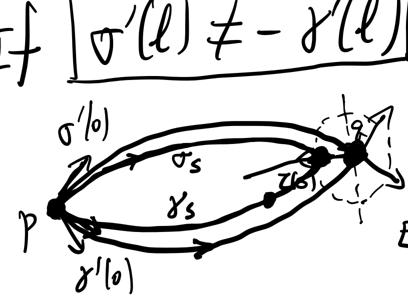
$\boxed{\exists\ \sigma \neq \gamma \quad \text{s.t.} \quad l(\sigma) = l(\gamma) = d(p,q)}$

Assume 1 does not hold. We need to show $\boxed{\sigma'(l) = -\gamma'(l)}$.

$\Rightarrow$ 2. hold.



If $\boxed{\sigma'(l) \neq -\gamma'(l)} \Rightarrow \exists\ v \in T_qM$, $\boxed{\begin{array}{l} \langle v, \gamma'(l)\rangle < 0 \\ \langle v, \sigma'(l)\rangle < 0 . \end{array}}$



$\tau(0) = q$, $\tau'(0) = v$

$\mathrm{Exp}_p : \underset{\underset{l\cdot\gamma'(0)}{\cup}}{U} \longrightarrow W \subset M$ is a diffeomorphism.

$\Rightarrow \tau(s) = \mathrm{Exp}_p(\underline{v(s)})$ $\quad \underline{v(s) \in U}$. $\quad \dfrac{v(s)}{l} \sim \gamma'(0)$

$\gamma_s(t) = \mathrm{Exp}_p\left(\frac{t}{l} v(s)\right)$. $\gamma_s(l) = \mathrm{Exp}_p(v(s)) = \tau(s)$.

$$\boxed{\frac{d}{ds} L(\gamma_s) < 0 \qquad \frac{d}{ds} L(\sigma_s) < 0}$$

$$\left(\begin{aligned} &\frac{1}{2}\frac{d}{ds} \underset{\shortparallel}{E(\gamma_s)} = \langle V, \gamma' \rangle \Big|_0^l = \langle V(l), \gamma' \rangle(l) < 0 \\ &\frac{d}{ds}\frac{1}{2}\frac{1}{l} L(\gamma_s)^2 = \frac{dL}{ds} \end{aligned} \right)$$

If $L(\gamma_s) = L(\sigma_s)$, then q is not the closest cut-pt. to p.

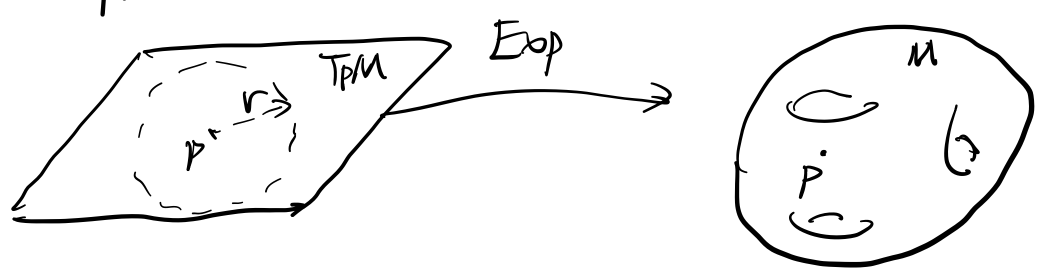
If $L(\gamma_s) < L(\sigma_s)$ then $\tau(s)$ is not the cut point of p along σ_s

$\Rightarrow$ The cut point of $\underset{\substack{\shortparallel \\ p}}{\sigma(0)}$ along σ_s has distance to p less

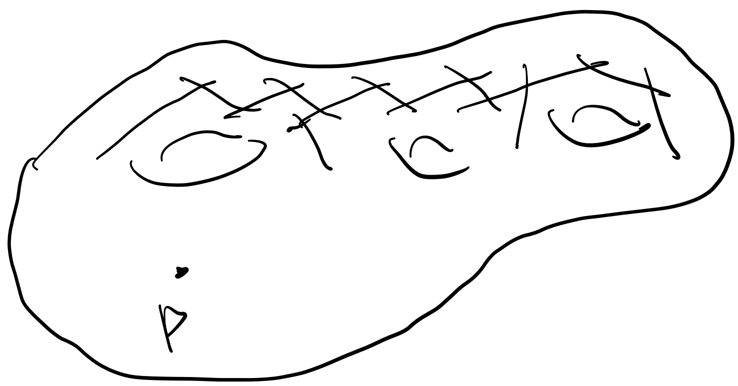
then $L(\sigma_s) < l$. ■

- For any $r < \mathrm{inj}_p$,

$\mathrm{Exp}_p : B_r(0) \longrightarrow M$ is a diffeomorphism.



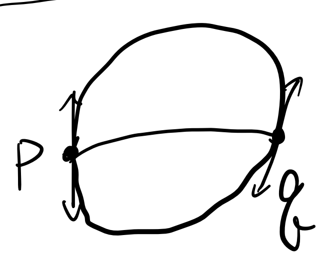
- For any $q \in M \setminus C_m(p)$, there is a unique minimizing geodesic from p to q.



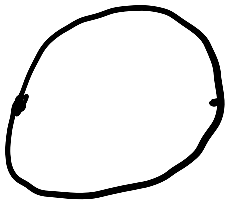
$$\mathrm{Exp}: \begin{array}{l} U \xrightarrow{\text{diffeomorphism}} M \setminus C_m(p) \\ \cap \\ T_pM \end{array}$$

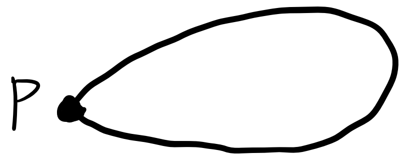
$$\inf \operatorname{inj} p = \inf_p \underbrace{d(p, C_m(p))}_{\substack{\| \\ \inf_v \underbrace{f(p, v)}_{v \in S^{n-1} \subset T_pM}}} = \operatorname{inj} M$$

$$\boxed{\text{If } M \text{ is compact, then } \exists\, p, q \in M \text{ s.t. } d(p,q) = \operatorname{inj} M}$$



1. $\exists$ minimizing geodesic γ s.t. p, q conjugate along γ
2. closed geodesic passing through p and q and any other geodesic loop has a longer length.





$$\{\text{closed geodesic}\} \subseteq \{\text{geodesic loop}\}$$