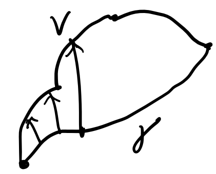
Morse Index Thm.

$\gamma: [0,a] \to M$ geodesic

$$\mathcal{V}(a) = \mathcal{V} = \left\{ V : \begin{array}{l} \text{piecewise differentiable vector fields along } \gamma,\ V(0)=0=V(a) \\ \langle V, \gamma'(t)\rangle = 0. \text{ (normal v.f.)} \end{array} \right.$$



$$\frac{1}{2}\frac{d^2}{ds^2}E(c_s) = I_a(V,V) = \int_0^a \left(\langle V', V' \rangle - \langle R(\gamma', V)V, \gamma' \rangle \right) dt$$

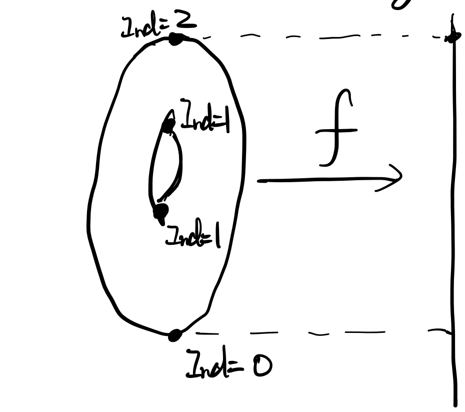
$$\mathrm{Ind}(I_a) = \max\left\{ \dim W : \text{subspace } W \subset \mathcal{V},\ I_a|_W \text{ is negative definite} \right\}$$

<u>Thm:</u> $\displaystyle \mathrm{Ind}(I_a) = \sum_{\substack{P_i \text{ conjugate points} \\ \text{for } \gamma(0) \text{ along } \gamma}} \mathrm{mult}(P_i) < +\infty .$

$$P_i = \gamma(t_i^*) : \mathrm{mult}(P_i) = \dim\{ J : \text{Jacobi field along } \gamma,\ J(0)=0=J(t_i^*) \}$$
$$= \dim \ker \left((d\exp_{\gamma(0)})_{t_i^*\gamma'(0)} : T_{\gamma(0)}M \to T_{\gamma(t_i^*)}M \right)$$
$$= \dim \mathrm{Null}(I_{t_i^*}) .$$

$$\mathrm{Null}(I_t) = \{ V \in \mathcal{V}(t) : I_t(V, W) = 0,\ \forall W \in \mathcal{V}(t) \}$$

Motivation: Morse Theory



Nondegenerate critical point P:

$\mathrm{Hess} f|_P$ is nondegenerate

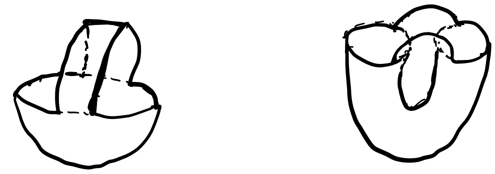
$\Downarrow$

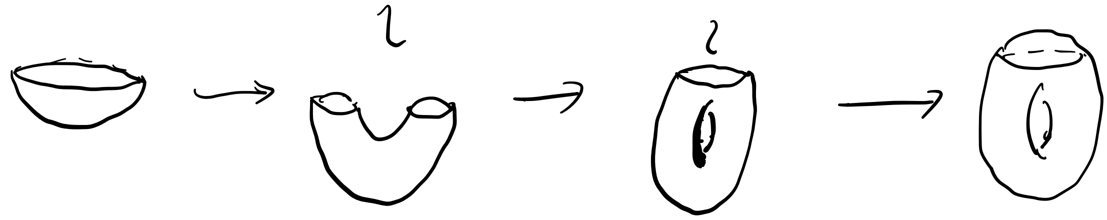
$$\mathrm{Hess} f|_P \underset{\sim}{\text{diagonalizable}} \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_r & & & & & \\ & & & \underbrace{0 \; \cdots \; 0}_{d} & & & & \\ & & & & \lambda_{r+d+1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_{r+d+s} \end{pmatrix}$$

$$\lambda_1 \le \cdots \le \lambda_r < 0 = \cdots = 0 < \lambda_{r+d+1} \le \cdots \le \lambda_{r+d+s}$$

$$r = \mathrm{Ind}(\mathrm{Hess} f|_P) = \mathrm{Ind}(P)$$

The sublevel set $M^{\le t} = f^{-1}((-\infty, t])$ changes topology as t crosses any critical point: up to homotopy, by attaching a r-handle if $\mathrm{Ind} = r$.





Indices of critical points $\rightsquigarrow$ topology of M

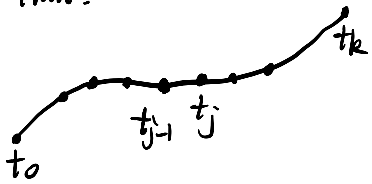
$\rightsquigarrow$ Indices of geodesics $\rightsquigarrow$ topology of $\Omega_{p,q}$ $\rightsquigarrow$ topology of M

geodesics $=$ critical points of energy functional; $\Omega_{p,q} = \{$curves connecting p to $q\}$

$$E: \Omega_{p,q} \longrightarrow \mathbb{R}$$

Proof of Morse Index Thm:

subdivide



$\gamma|_{[t_{j-1},t_j]}$ minimizing

contained in a (totally) normal nbhd.

$$\mathcal{V}^+ = \{ V \in \mathcal{V} : \quad V(t_j) = 0, \quad j = 0, \cdots, k.$$

$$\mathcal{V}^- = \{ V \in \mathcal{V} : \quad \underline{V|_{[t_{j-1}, t_j]}} \text{ are Jacobi fields} \}$$

broken Jacobi field $\qquad$ determined by $V(t_{j-1})$ and $V(t_j)$

$$\mathcal{V}^- \cong \bigoplus_{j=1}^{k-1} T_{\gamma(t_j)} M \quad \text{finite dimensional.}$$

- $I_a|_{\mathcal{V}^+}$ is positive definite. (use $\gamma|_{[t_{j-1},t_j]}$ is energy minimizing)

  $I_a(\mathcal{V}^+, \mathcal{V}^-) = 0$

$$\Rightarrow \quad \mathrm{Ind}(I_a) = \mathrm{Ind}(I_a|_{\mathcal{V}^-}) < +\infty.$$

- Define $\quad i(t) = \mathrm{Ind}(I_t|_{\mathcal{V}(t)})$

  Then: $\qquad t \le \bar{t} \Rightarrow i(t) \le i(\bar{t})$

$$i(t-\varepsilon) = i(t) \quad \text{for } 0 < \varepsilon \ll 1$$

(continuity of I_t w.r.t. $t \Rightarrow I_t|_W$ neg. def. $\Rightarrow I_{t-\varepsilon}|_W$ neg. def.)

- Claim: $i(t+\varepsilon) = i(t) + d$

$$d = \dim \mathrm{Null}(I_t|_{\mathcal{V}(t)}) = \mathrm{mult}(\gamma(t)).$$

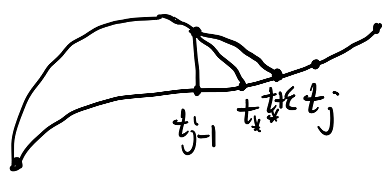
Proof of Claim: $\underset{\displaystyle \mathcal{V}^-(t)}{I_t|_{\underset{\cap}{W}}}$ positive definite $\Longrightarrow$ $\underset{\displaystyle \mathcal{V}^-(t+\varepsilon)}{I_{t+\varepsilon}|_{\underset{\cap}{W}}}$ is positive def.

$$\Longrightarrow \quad \dim \mathcal{V}^-(t) - \big(i(t)+d\big) \leq \underbrace{\dim \mathcal{V}^-(t+\varepsilon) - i(t+\varepsilon)}_{\substack{\text{positive def.} \oplus \text{null space} \\ \text{subspace}}}$$

$$\Longrightarrow \quad i(t+\varepsilon) \leq i(t)+d.$$

For the other direction, we need to produce new vector fields from the null space of I_t to contribute to negative space of $I_{t+\varepsilon}$.

Choose any $V \in Null(I_{t_*})$ with $t_* \in (t_{j-1}, t_j)$. Define



$$W(t) = \begin{cases} V(t) & 0 \leq t \leq t_{j-1} \\ J_2(t) & t_{j-1} \leq t \leq t_* + \varepsilon \end{cases}$$

J_2: Jacobi field on $[t_{j-1}, t_*+\varepsilon]$ s.t. $J_2(t_{j-1}) = V(t_{j-1})$, $J_2(t_*+\varepsilon) = 0$.

Then $\overset{0}{\shortparallel}$

$$I(V,V) = \int_0 \underbrace{\big(\langle V', V'\rangle - \langle R(\gamma', V)V, \gamma'\rangle\big)}_{B(V,V)} dt$$

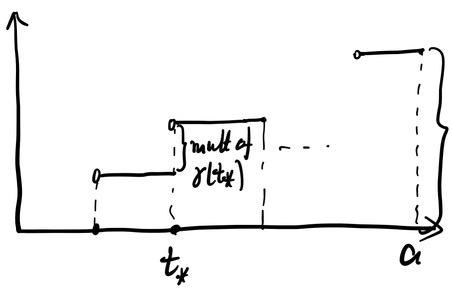
$$= \int_0^{t_{j-1}} \underbrace{B(V,V)}_{B(W,W)} dt + \underbrace{\int_{t_{j-1}}^{t_*+\varepsilon} B(V,V)\, dt}_{\displaystyle \vee \atop \displaystyle \int_{t_{j-1}}^{t_*+\varepsilon} B(\underset{W}{J_2}, \underset{W}{J_2})\, dt} > I_a(W,W)$$

Index Lemma.

Check the fields we get from negative subspace of $I_t|V^-(t)$ and $\mathrm{null}(I_t|V^-(t))$ are linearly independent.

$$\Rightarrow \quad i(t+\varepsilon) \geq i(t)+d.$$

So we get the graph of $i(t)$:



$$= \mathrm{Ind}(I_a) = \sum_{\substack{\gamma(t_i^*) \\ \text{conjugate to } \gamma(0)}} \mathrm{mult}(\gamma(t_i^*))$$

■

Cut point of $P=\gamma(0)$ along γ: $\gamma(t_0)$ where

$$t_0 = \underset{\text{max}}{\overset{\shortparallel}{\sup}} \left\{ t : \gamma|[0,t] \text{ is distance minimizing} \right\}$$

$$\Updownarrow$$

$$\ell(\gamma|_{[0,t]}) = \mathrm{dist}(\gamma(0), \gamma(t))$$

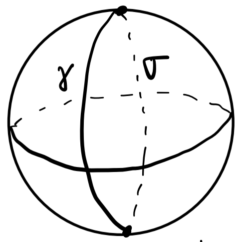
Prop: The cut point $\gamma(t_0)$ satisfies one of the following 2 conditions.

Either (i) $\gamma(t_0)$ is the 1st. conjugate point of $\gamma(0)$ along γ.

OR (ii) There exists geodesic $\sigma \neq \gamma$ that connects $\gamma(0)$ to $\gamma(t_0)$.

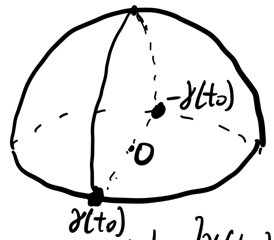
Ex:

S^n



$\gamma(t_0) =$ 1st conj. point

$\mathbb{R}P^n = S^n / \{p \sim (-p)\}$



cut point $[\gamma(t_0)] = [-\gamma(t_0)]$

1st. conjugate point $= \{[N] = [-N]\}$ of $N = \gamma(0)$

If $\sec \leq 0$, then no conjugate points along γ, so the 2nd condition always happen.