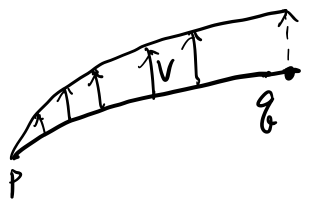


$\gamma: [0,a] \to M$.

V: piecewise diff. vector field along γ. $V(0)=0$

$\langle V, \gamma' \rangle = 0$

$$I_a(V,V) = \int_0^a (\langle V', V' \rangle - \langle R(\gamma', V)V, \gamma' \rangle)\, dt.$$

Index Lemma: If γ has no conjugate point, then for any V

$\boxed{I_a(V,V) \geq I_a(J,J)}$ where J is a normal Jacobi field along γ

s.t. $J(0)=0$. $J(a)=V(a)$. $\langle J, \gamma' \rangle = 0$.

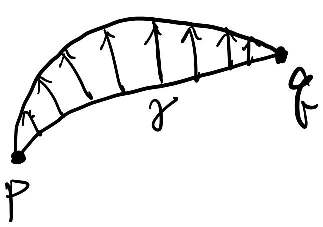
$$\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} = I_a(V,V) \geq I_a(J,J) = \frac{1}{2}\frac{d^2}{ds^2}E^J(s)\Big|_{s=0}$$

$$\frac{1}{2}\frac{d^2}{ds^2}E(c_s) \qquad \Uparrow \qquad \frac{1}{2}\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}.$$

$$\boxed{E(c_s) \geq E(\gamma_s)}$$

geodesics

$$\frac{d}{ds}E(s)\Big|_{s=0} = \langle \gamma', V \rangle\Big|_0^a = 0 = \frac{d}{ds}E^J(s).$$



$$\boxed{\mathcal{V} = \left\{ \begin{array}{l} V \text{ piecewise diff. } \langle V, \gamma' \rangle = 0 \\ V(0) = V(a) = 0. \end{array} \right\}}$$

Index of I_a = $\max\{\dim W : W \subseteq \mathcal{V}.\ I_a|_W$ is negative definite$\}$

i.e. $I_a(V,V) < 0,\ \forall V \in W$

Symmetric bilinear form

Nullity of $I_a = \dim \underbrace{\{V \in \mathcal{V} : I_a(V,W)=0,\ \forall W \in \mathcal{V}\}}_{= N(I_a)}$

Finite Dim.: A $n \times n$ symmetric

$\rightsquigarrow$ A is diagonalizable by an orthogonal matrix S

$$S^{-1} A \cdot S = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\underline{\lambda_1 \le \cdots \le \lambda_q} < 0 = \lambda_{q+1} = \cdots = \lambda_{q+\underline{k}} < \lambda_{q+k+1} = \cdots = \lambda_n$$

$q = \text{index}(A)$. $\dim N(A) = k$.

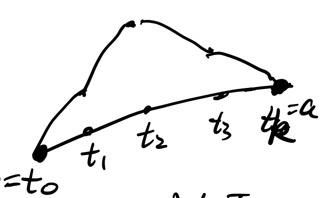
__Prop:__ If $V \in \mathcal{V}$ is in $\underline{\underset{\text{null space}}{N(I_a)}}$ $\Leftrightarrow$ V is a Jacobi field along γ

$\Downarrow$

$(V$ is smooth$)$

__Pf:__ "$\Leftarrow$" $\underline{V = J}$. $\underline{\forall W \in \mathcal{V}}$

$$\underline{I_a(V,W)} = \int_0^a \Big(\underset{\frac{d}{dt}\langle V',W\rangle - \langle V'',W\rangle}{\underline{\langle V',W'\rangle}} - \underset{+\langle R(\gamma',V)\gamma',W\rangle}{\underline{\langle R(\gamma',V)W,\gamma'\rangle}}\Big)\,dt$$



$V = J$

$$= -\int_0^a \underset{= 0}{\underline{\langle V'' - R(\gamma',V)\gamma', W\rangle}}\,dt + \underset{= \sum_{j=0}^{k} \langle V', W\rangle\big|_{t_j^+}^{t_{j+1}^-}}{\underline{\langle V', W\rangle\big|_0^a}} = 0$$

"$\Rightarrow$" If for any $W \in \mathcal{V}$,

$$I_a(V,W) = -\int_0^a \langle V'' - R(\gamma', V)\gamma', W\rangle \, dt + \sum_{j=0}^{k} \langle V', W \rangle \Big|_{t_j^+}^{t_{j+1}^-} = 0$$

$\Rightarrow$ $V'' - R(\gamma', V)\gamma' = 0$ and V is smooth at t_j

$\Rightarrow$ V is a Jacobi field. ∎

$$\dim N(I_a) = \dim \{ \overset{(d\exp_p)_{t\gamma'(0)}(t\cdot w)}{\overset{\parallel}{J}} : \text{Jacobi field}, \ J(0) = 0 = J(a) \}$$

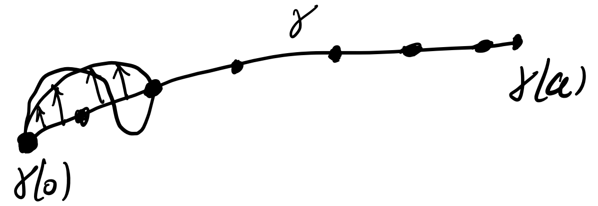
$$= \dim \ker\left((d\exp_p)_{a\cdot\gamma'(0)} : T_pM \to T_qM \right)$$

$$= \text{multiplicity of } q \text{ along } \gamma.$$

q is a conjugate point $\Leftrightarrow$ $N(I_a) \neq \{0\}$

multiplicity of q $=$ $\dim N(I_a)$.

Thm (Morse Index Thm) Index of I_a is finite and is equal to the number of points $\gamma(t)$ $0 < t < a$, conjugate to $\gamma(0)$ counted with multiplicity.



Cor: The set of conj. points along any geodesic is discrete.

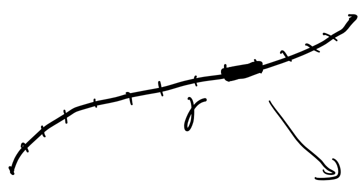
Cor: If there is a conjugate point $\gamma(t_0)$, $0<t_0<a$ then γ can not be minimizing.

(**Because:** $\mathrm{Ind}(I_a)>0$: $\exists\ V\in\mathcal{V}$ s.t. $\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0}<0$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad I_a(V,V)$

$\Rightarrow \gamma$ can't be energy minimizing

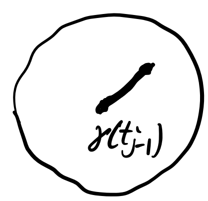
$\Rightarrow \gamma$ is not distance minimizing.)

Proof of Morse Index Thm:



$T_{\gamma(t_{j-1})}M$

$\big\downarrow \exp_{\gamma(t_{j-1})}$ $\boxed{\text{diffe.}}$



subdivision

$0=t_0<\underline{t_1<\cdots<t_{k-1}}<t_k=a$. s.t.

- $\gamma|_{[t_{j-1},t_j]}\subset U_j\subset \exp_{\gamma(t_{j-1})}(B_\delta(0))$
- $\gamma|_{[t_{j-1},t_j]}$ is minimizing and has no conjugate points w.r.t. $\gamma(t_{j-1})$

$$\mathcal{V}^-=\{V\in\mathcal{V}:\ V|_{[t_{j-1},t_j]} \text{ is a Jacobi field } \forall j=1,\dots,k\}\cong(\underset{\substack{\text{|||}\\ T_{\gamma(t_j)}M}}{\mathbb{R}^{n-1}})^{k-1}$$

$\uparrow$ determined by $V(t_{j-1})$ and $V(t_j)$.

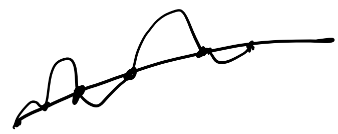
$$\left\{\begin{cases} V''-R(\gamma',V)\gamma'=0\\ V(t_{j-1})=W_1\\ V'(t_{j-1})=W_2\end{cases}\right\}\longleftrightarrow \underline{\{V(t_{j-1}),V(t_j)\}}$$

$\uparrow$ no conjugate points.

$\mathcal{V}^+ = \{ V \in \mathcal{V} : \underline{V(t_j) = 0, \ j = a, \cdots, k} \}.$

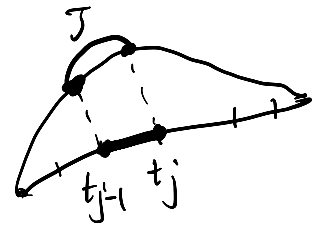
$V(t_0) = V(0) = 0$

$V(t_k) = V(a) = 0.$



$V \in \mathcal{V} \rightsquigarrow \exists$ unique $W \in \mathcal{V}^-$ s.t. $W(t_j) = V(t_j)$

$\Rightarrow V - W \in \mathcal{V}^+.$



$$\left.\begin{array}{l} V = \underset{\mathcal{V}^-}{\underset{\cap}{W}} + \underset{\mathcal{V}^+}{\underset{\cap}{(V-W)}} \\ \\ \mathcal{V}^- \cap \mathcal{V}^+ = \{0\} \end{array}\right\} \Rightarrow \underline{\mathcal{V} = \mathcal{V}^- \oplus \mathcal{V}^+}$$

$\mathcal{V}^- \cong (\mathbb{R}^{n-1})^{k-1}$

<u>Prop</u>: • $\mathcal{V}^- \perp_{I_a} \mathcal{V}^+$: $I_a(W_1, W_2) = 0 \quad \forall\, W_1 \in \mathcal{V}^-, \ W_2 \in \mathcal{V}^+$

$W_1 \in \mathcal{V}^-$, $W_2 \in \mathcal{V}^+$

• $\underline{I_a|_{\mathcal{V}^+} \text{ is positive definite.}}$

$\Rightarrow \operatorname{Ind}(I_a) = \underline{\operatorname{Ind}(I_a|_{\mathcal{V}^-})} = \max\{\dim \mathcal{W} : \mathcal{W} \subseteq \mathcal{V}, \ \underline{I_a|_{\mathcal{W}} \text{ neg. def.}}\}$

$\Leftarrow \mathcal{W} \cap \mathcal{V}^+ = 0$

$\Downarrow$

So $\dim \pi^-(\mathcal{W}) = \dim \mathcal{W}$

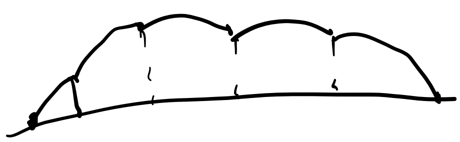
$I_a(W_1, W_1) \Leftarrow \pi^- : \mathcal{W} \to \mathcal{V}^-$ is injective

$W_1 + W_2 \mapsto W_1$

$I_a(W_1, W_1) = I_a(W, W) - I_a(W_2, W_2) < 0$

$$I_a(W_1,W_2)=\int_0^a \Big(\underbrace{\langle W_1',W_2'\rangle}_{\frac{d}{dt}\langle W_1',W_2\rangle-\langle W_1'',W_2\rangle} - \langle R(\gamma',W_1)W_2,\gamma'\rangle\Big)\,dt$$

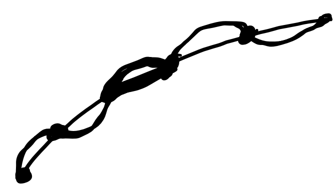
$$= -\int_0^a \underbrace{\langle W_1''-R(\gamma',W_1)\gamma',W_2\rangle}_{0}\,dt + \sum_{j=0}^{k-1}\langle W_1',W_2\rangle\Big|_{t_j^+}^{t_{j+1}^-}$$



$= 0$. if $\underset{\mathcal{V}^-}{\underset{\cap}{W_1}}$ broken Jacobi and $\underset{\mathcal{V}^+}{\underset{\cap}{W_2(t_j)=0}}$.

Choose $V\in\mathcal{V}^+$. $\underline{V(t_j)=0}$.

$$I_a(V,V)=\frac{1}{2}\frac{d^2}{ds^2}\underline{E(c_s)}\underset{\text{"}>0\text{"}}{\geq} 0.$$



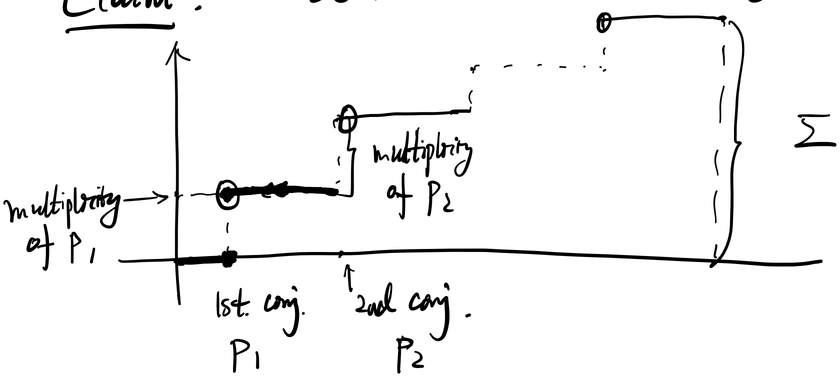
∎

Just need to calculate $\operatorname{ind}\big(I_a|_{\mathcal{V}^-}\big)$, $\mathcal{V}^- = \{\text{broken Jacobi fields}\}$.

- Define $i(t)=$ Index of I_t.

For $0 < t \ll 1$, $i(t) = 0$.

<u>Claim</u>: $i(t)$ is an <u>increasing function</u>



$\sum$ multiplicities of conjugate pts.

$$\lim_{t \to t_*^-} i(t) = i(t_*)$$

- If $\bar{t} > t$, then $i(\bar{t}) \geq \underset{\max\{\dim W \,:\, W \subset \mathcal{V}_t^-,\ \underline{I_t|_W \text{ neg.}}\}}{\underset{\shortparallel}{\underline{i(t)}}}$.

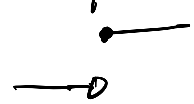
$I_t(V) < 0 \Rightarrow I_{\bar{t}}(\bar{V}) = I_t(V) < 0$

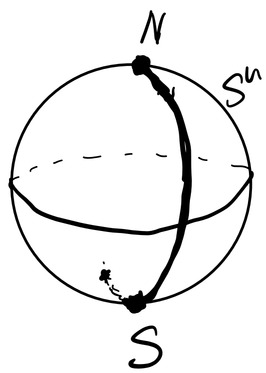
extend V by 0 to $\bar{t}$

- Left continuous. If $0 < \varepsilon \ll 1$, then $\underline{i(t-\varepsilon) = i(t)}$.

$\downarrow$

$\uparrow$





$\underline{\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} \geq 0}$.

$\dim N(I_a) = \underline{n-1} = $ multiplicity of S