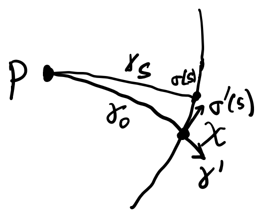


$r(x) = \mathrm{dist}(x, p) = d(x, p)$

$r(s) = r(\sigma(s)) = d(\sigma(s), p) = \ell(\gamma_s)$

$\gamma_s : [0,1] \to M$ minimizing geodesic $p \to \sigma(s)$

$r' = \frac{d}{ds} r$, $r'' = \frac{d^2}{ds^2} r$. $E(s) = E(\gamma_s) = \int_0^1 |\gamma'(s)|^2 ds = \ell(\gamma_s)^2 = r(s)^2$

$\frac{1}{2} E' = \frac{1}{2}\frac{d}{ds} E = r \cdot r'$, $\underline{\frac{1}{2} E'' = r'^2 + r \cdot r''}$

$\Rightarrow r' = \frac{1}{r} \cdot \frac{1}{2} E'$, $\underline{r'' = \left(\frac{1}{2} E'' - r'^2\right) \cdot \frac{1}{r}}$

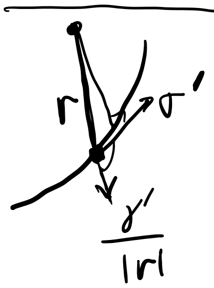
$\frac{1}{2} E' = \frac{1}{2} \frac{d}{ds} \int_0^1 |\gamma'(t)|^2 dt = \int_0^1 \langle \nabla_{\partial_s} \partial_t, \partial_t \rangle dt = \langle \partial_s, \partial_t \rangle \Big|_0^1 = \underline{\langle \sigma', \gamma' \rangle}$

$\langle \nabla_{\partial_s} \partial_t, \partial_t \rangle = \langle \nabla_{\partial_t} \partial_s, \partial_t \rangle = \underline{\partial_t \langle \partial_s, \partial_t \rangle} - \langle \partial_s, \nabla_{\partial_t} \partial_t \rangle$, $\nabla_{\partial_t} \partial_t = 0$

$\underline{\Rightarrow r' = \frac{1}{r} \langle \sigma', \gamma' \rangle = \langle \sigma', \frac{\gamma'}{r} \rangle}$, $\sigma' = J(1)$

$\gamma : [0,1] \to M$

$\int_0^1 \underline{|\gamma'(t)|} dt = |\gamma'| \cdot 1 = r$



$J = \frac{\partial}{\partial s} \gamma(s,t) \Big|_{s=0} = J$, $\underline{J(0) = 0}$

$\frac{1}{2} E'' = \int_0^1 \left(\underline{\langle J', J' \rangle} - \underline{\langle R(\gamma', J) J, \gamma' \rangle} \right) dt + \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \gamma' \rangle \Big|_0^1$

$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0$

$\langle J', J' \rangle = \underline{\frac{\partial}{\partial t} \langle J', J \rangle} - \langle J'', J \rangle$

$\langle J'', J \rangle = \underline{\langle R(\gamma', J) \gamma', J \rangle}$

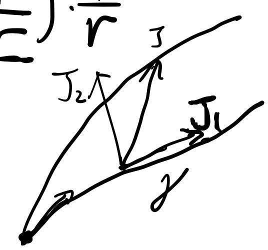
$= \langle J', J \rangle \Big|_0^1 = \langle J'(1), J(1) \rangle = E'' \cdot \frac{1}{2}$, $J(1) = \sigma'$

$$J(1) = \frac{\partial}{\partial s} f(s,1)\Big|_{s=0} = \frac{\partial}{\partial s}\sigma(s)\Big|_{s=0} = \sigma'(\underset{\substack{\uparrow \\ s}}{0}).$$

$$J(t) = (d\exp_p)_{t\gamma'(0)}(t\cdot w). \quad \underline{J(1) = (d\exp_p)_{\gamma'(0)}(w) = \sigma'}$$

$$r'' = \left(\frac{1}{2}E'' - \underline{r'^2}\right)\cdot\frac{1}{r}$$

$$= \underline{\langle J'(1), J(1)\rangle\cdot\frac{1}{r}} - \Big(\underline{\overset{\substack{|J_1|^2 \\ \parallel}}{\langle J(1), \gamma'\rangle}\cdot\frac{1}{r}}\Big)^2\cdot\frac{1}{r}$$



$$J(t) = J_1 + J_2 \qquad J_2 \perp \gamma'$$

$$= a\cdot\underline{t\cdot\gamma'(t)} + J_2 . \quad J(1) = a\cdot\gamma'(1) + J_2(1)$$

$$J' = a\cdot\gamma'(t) + J_2' . \quad J'(1) = a\cdot\gamma'(1) + J_2'(1)$$

$$\underline{\langle J', J\rangle(1) = |J_1(1)|^2 + \langle J_2', J_2\rangle}$$

$$\boxed{r'' = \frac{1}{r}\left(|J_1(1)|^2 + \langle J_2', J_2\rangle\right) - \frac{1}{r}|J_1(1)|^2 = \underline{\frac{1}{r}\langle J_2', J_2\rangle}}$$

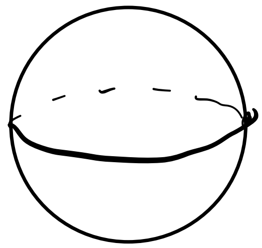
$$r' = \frac{1}{r}\cdot\langle \underset{\substack{\parallel \\ J_1+J_2}}{J(1)}, \gamma'(1)\rangle = \langle \underline{J_1}, \gamma'(1)\rangle\frac{1}{r}$$

$$\boxed{M_\lambda = \text{space form with constant sectional curvature}}$$

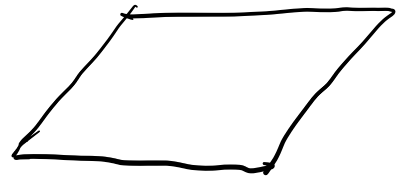
$$\underbrace{\tilde{J}_2(t) = J_2^\lambda(t) = sn_\lambda(\underline{r}\cdot t)\cdot w(t)}_{} \begin{cases} \sin(r\cdot t)\, w(t) & \lambda = 1 \\ \underline{r\cdot t}\cdot w(t) & \lambda = 0 \\ sh(r\cdot t)\, w(t) & \lambda = -1. \end{cases}$$

$J_2(r) \in T_{\gamma(1)}M$. $r = d(p, \gamma(1))$

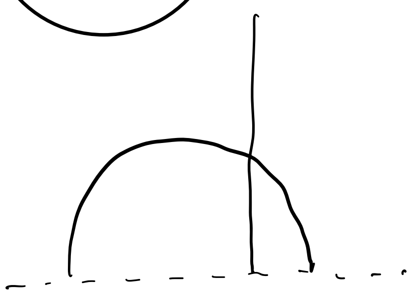
$\lambda = 1$



$\lambda = 0$



$\lambda = -1$



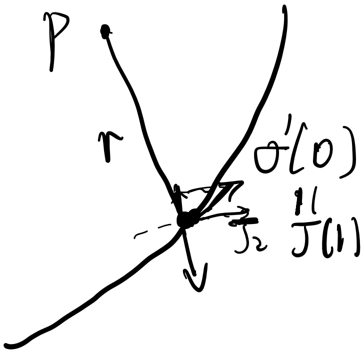
$$\frac{dx^2 + dy^2}{y^2} = g_{hyp}.$$

$$\underline{\langle J_2'(1), J_2(1)\rangle} = \langle r\cdot sn_\lambda'(r\cdot)\cdot w(t),\ sn_\lambda(r\cdot)\, w\rangle$$

$$= r\cdot\frac{sn_\lambda'(r)}{sn_\lambda(r)} \underbrace{\langle sn_\lambda(r)\, w(t),\ sn_\lambda(r)\, w\rangle}_{= |J_2(1)|^2}$$

$$\boxed{r'' = \frac{1}{r}\langle J_2', J_2\rangle = \frac{sn_\lambda'(r)}{sn_\lambda(r)}\cdot |J_2(1)|^2}$$

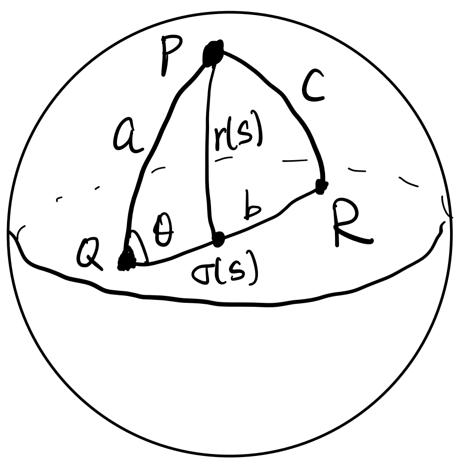
$J = J_1 + J_2$. $J(1) = \sigma'(s)\big|_{s=0}$



$$\boxed{r' = \frac{1}{r}\langle \sigma', \gamma' \rangle = \langle \underbrace{J(1)}_{J_1 + \underline{J_2}}, \frac{\gamma'}{r} \rangle}$$

$$\underbrace{c \cdot \frac{\gamma'}{r}}_{\|}$$

$$r' = \langle \underline{J_1}, \frac{\gamma'}{r} \rangle$$



$$r(s) = d(P, \sigma(s))$$

$$\underline{r(0) = d(P, Q) = a}$$

$$r(b) = d(P, R) = c.$$

$$\boxed{\varphi(s) = h(r(s))}.$$

$$h : [0, +\infty) \to \mathbb{R}.$$
$$r \mapsto h(r)$$

$$\underline{\varphi' = h_r \cdot r'}$$

$$\varphi'' = h_{rr} (r')^2 + h_r \cdot r''$$

$$= \underbrace{h_{rr}}_{b(r)} \cdot \underline{|J_1|^2} + \underbrace{h_r \cdot \frac{sn'_\lambda(r)}{sn_\lambda(r)}}_{b(r)} \cdot \underline{|J_2|^2}$$

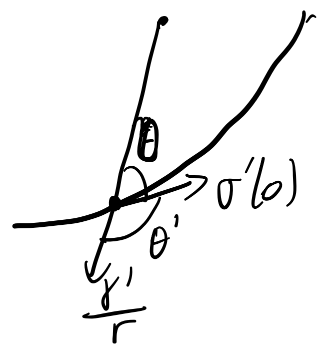
$$= \underline{b(r) \cdot \underbrace{(|J_1|^2 + |J_2|^2)}_{|J|^2 = |\sigma'(s)|^2} = b(r) \,. = \pm\varphi}$$

$$\underline{h_{rr} = h_r \cdot \frac{sn_\lambda'(r)}{sn_\lambda(r)}} \qquad sn_\lambda(r) = \begin{cases} \sin(r) & \lambda = 1 \\ r & \lambda = 0 \\ sh(r) & \lambda = -1. \end{cases}$$

$\lambda = 1$: $\boxed{h = \cos(r)},$

$$\underline{h_{rr} = -\cos(r)}, \quad h_r \cdot \frac{\sin'(r)}{\sin(r)} = \underline{-\cos(r) = h_{rr}}.$$

$$\boxed{\varphi'' = -\cos(r(s)) = -\varphi(s)}$$



$$\begin{cases} \boxed{\varphi'' + \varphi = 0} \\ \underline{\varphi(0) = \cos(r(0)) = \cos a} \\ \varphi'(0) = -\sin(r) \cdot \underset{\substack{\parallel \\ \langle J(1), \frac{\gamma'(1)}{r} \rangle = \cos\theta'}}{r'} = -\sin(a) \cdot \underset{\substack{\parallel \\ \sin(a)\cdot\cos\theta}}{\cos(\pi - \theta)} \end{cases}$$

$$\underline{\overset{\cos(r(s))}{\overset{\parallel}{\varphi(s)}} = C_1 \cdot \cos(s) + C_2 \cdot \sin(s).} \qquad \underline{r(s) = \cos^{-1}(\varphi(s))}$$

$$\cos(r(0)) = \varphi(0) = C_1 = \underline{\cos a}$$

$$\varphi'(0) = C_2 = +\sin(a) \cdot \cos\theta$$

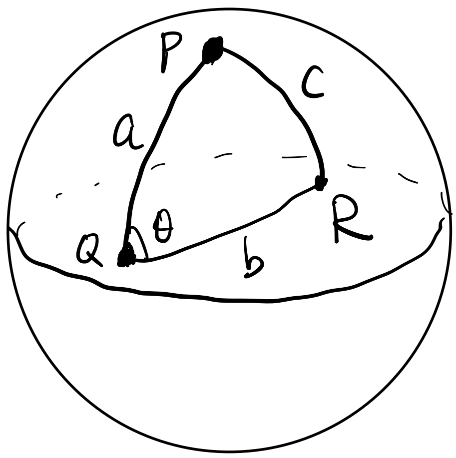
$$\varphi(b) = \cos a \cdot \cos b + \sin(a) \cdot \sin b \cdot \cos\theta$$

$$\parallel$$

$$\cos(r(b)) = \cos(c)$$

$$\boxed{\cos(c) = \cos(a)\cdot\cos b + \sin(a)\cdot\sin(b)\cdot\cos\theta}\qquad \lambda=1$$

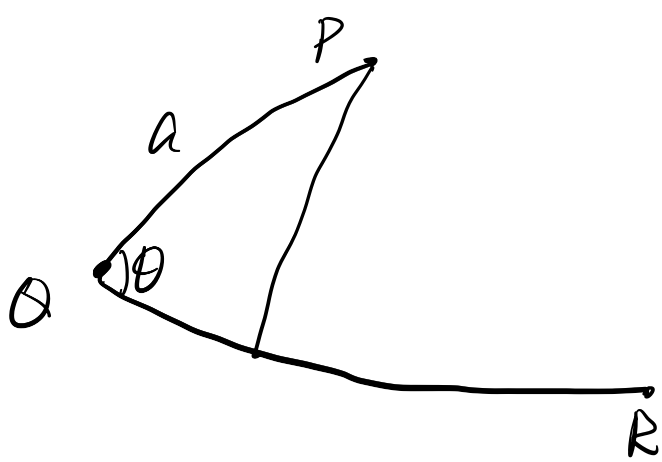
Law of Cosine.



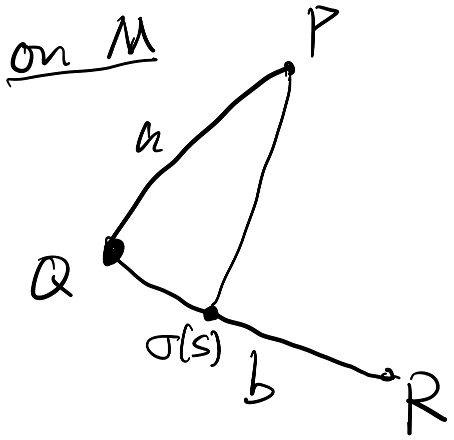
$$c^2 = a^2 + b^2 - 2\cdot a\cdot b\cdot\cos\theta \qquad \lambda=0$$

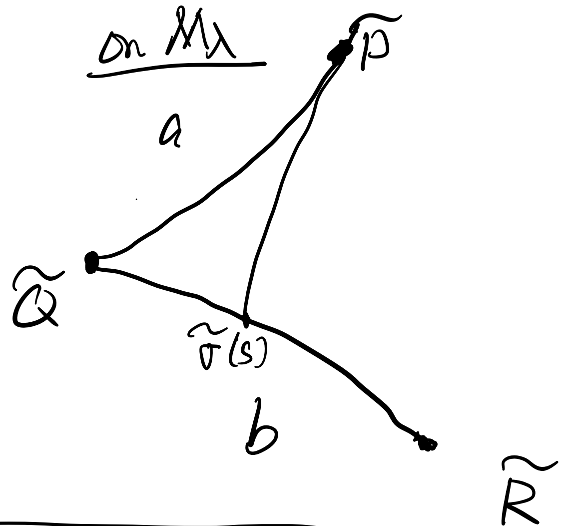
$$\mathrm{ch}(c) = \mathrm{ch}(a)\,\mathrm{ch}(b) - \mathrm{sh}(a)\,\mathrm{sh}(b)\cdot\cos\theta \qquad \lambda=-1$$

$$\mathrm{ch}(x) = \frac{e^x+e^{-x}}{2},\quad \mathrm{sh}(x) = \frac{e^x-e^{-x}}{2}.$$



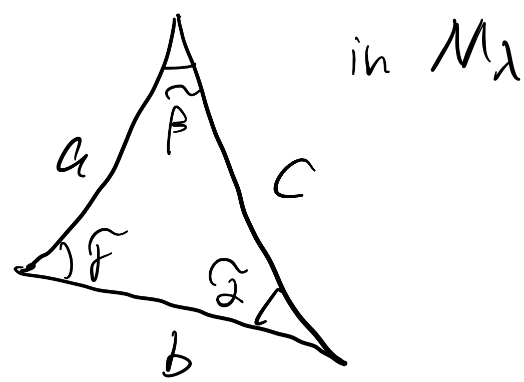
$\boxed{\text{Toponogov Thm}}$: $\sec_M \geq \underline{\lambda}$





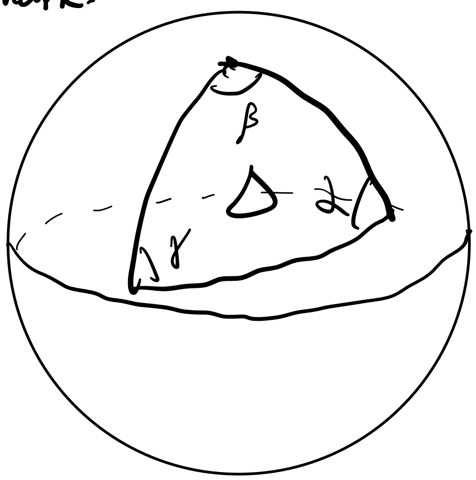
$$\boxed{d(P, \sigma(s)) \leq d(\tilde{P}, \tilde{\sigma}(s))}$$

Triangle Version $\sec_M \geq \lambda$

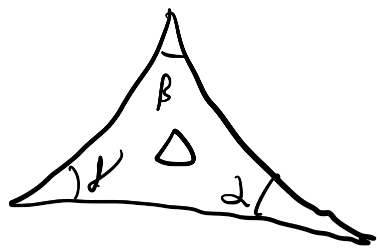


$$\Longrightarrow \quad \alpha + \beta + \gamma \geq \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}.$$

Remark:



$$\alpha+\beta+\gamma-\pi = \mathrm{Area}(\Delta)$$



$$\pi-(\alpha+\beta+\gamma) = \mathrm{Area}(\Delta)$$

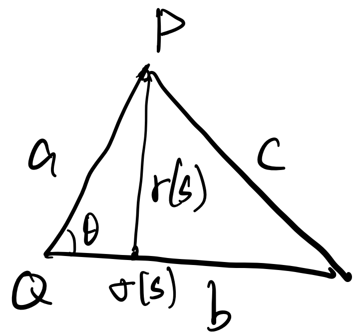
$\lambda=0$: $\quad h(r)=\frac{1}{2}r^2. \quad sn_0(r)=r$

$$\varphi(s)=h(r(s))=\frac{1}{2}\,r(s)^2$$

$$\varphi' = r\cdot r'$$

$$\varphi'' = \underbrace{r'^2}_{=|J_1|^2} + \underbrace{r\cdot r''}_{=r\cdot\frac{1}{r}\cdot|J_2|^2} = |J_1|^2+|J_2|^2=|\sigma'|^2=1$$

$$\varphi''=1 \Rightarrow \boxed{\varphi(s)=\frac{1}{2}s^2+c_1\cdot s+c_2}$$



$$\varphi(0) = \frac{1}{2} r(0)^2 = \frac{a^2}{2} = C_2$$

$$\varphi'(0) = a \cdot \langle \sigma', \gamma' \rangle = a \cos(\pi - \theta) = -\cos\theta = C_1$$

$$\underline{\varphi(s) = \frac{1}{2} s^2 - a \cdot \cos\theta \cdot s + \frac{1}{2} a^2 = \frac{1}{2} r(s)^2}$$

$\boxed{M. \quad \sec_M \geq 0}$.

$$\varphi(r(s)) = \frac{1}{2} r(s)^2 .$$

$$\varphi' = r \cdot r'$$

$$\varphi'' = r'^2 + r \cdot r'' = |J_1|^2 + \cancel{r} \cdot \frac{1}{\cancel{r}} \cdot \langle J_2', J_2 \rangle$$

$$= |J_1|^2 + \boxed{\langle J_2', J_2 \rangle}$$

$$\left(\text{on } M_0, \ \tilde{J}_2 = r \cdot t \cdot w \quad \langle \tilde{J}_2', \tilde{J}_2 \rangle = |J_2|^2 \right)$$

Index Lemma $\wedge$|

$\sec_M \geq 0$

$$\boxed{\langle \tilde{J}_2', \tilde{J}_2 \rangle}$$

$$I_1(J_2, J_2) = \langle J_2', J_2 \rangle = \int_0^1 \left(\langle J_2', J_2' \rangle - \underline{\langle R(\ ) \rangle} \right) dt$$

$$\wedge|$$

$$I_1(\tilde{J}_2, \tilde{J}_2) = \langle \tilde{J}_2', \tilde{J}_2 \rangle = \int \left(\langle \tilde{J}_2', \tilde{J}_2' \rangle - \underline{\langle R(\ ) \rangle} \right) dt$$

$$\left.\begin{array}{l}\varphi'' \leq 1. \\ \underline{\tilde{\varphi}'' = 1}\end{array}\right\} \xrightarrow[\text{condition}]{\substack{\text{same} \\ \text{Initial}}} \underset{\frac{1}{2}r(s)^2}{\underset{\shortparallel}{\varphi}} \leq \underset{\frac{1}{2}\tilde{r}(s)^2}{\underset{\shortparallel}{\tilde{\varphi}}}$$

$$\Rightarrow \quad r(s) \leq \tilde{r}(s).$$

$\lambda = -1$: $\quad \varphi(s) = ch(r(s))$. $h(r) = ch(r)$.

$$\Rightarrow \quad \varphi''(s) - \varphi(s) \leq 0.$$

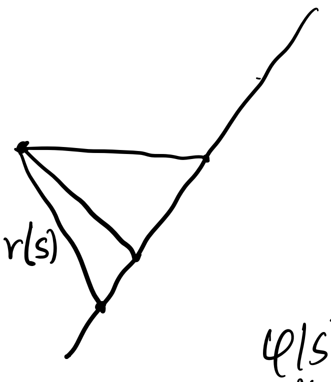
Index Lemma

on M_{-1}: $\tilde{\varphi}''(s) - \tilde{\varphi}(s) = 0$.

$$\xRightarrow[\text{condition}]{\text{same initial}} \quad \underset{ch(r(s))}{\underset{\shortparallel}{\varphi(s)}} \leq \underset{ch(\tilde{r}(s))}{\underset{\shortparallel}{\tilde{\varphi}(s)}}. \Rightarrow r(s) \leq \tilde{r}(s).$$

Similarly for $\lambda = 1$. $\varphi(s) = \cos(r(s))$.

On $\mathbb{R}^2$:



$$r(s) = |\sigma(s) - P|$$

$$= \sqrt{(as+b-x_0)^2 + (as+b-y_0)^2}$$

$$h(r) = \frac{1}{2}r^2 \Rightarrow \underset{\varphi(s)}{\frac{1}{2} r(s)^2} = \frac{1}{2}\left((\quad)^2 + (\quad)^2\right)$$