Rauch Comparison Thm

$\gamma : [0,a] \to M$ geodesics with same speed: $|\gamma'(t)| = |\tilde{\gamma}'(t)| = \text{const.}$

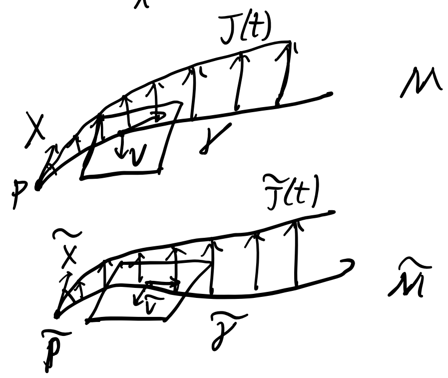
$\tilde{\gamma} : [0,a] \to \tilde{M}$

J : Jacobi field along γ : $J(0) = 0$, $\underline{\langle \overset{X}{\overset{\|}{J'(0)}}, \gamma'(0) \rangle}$ $\underline{|J'(0)|}$

$\tilde{J}$: . . . . $\tilde{\gamma}$ $\tilde{J}(0) = 0$, $\underset{\tilde{X}}{\underset{\|}{\underline{\langle \tilde{J}'(0), \tilde{\gamma}'(0) \rangle}}}$ $\underline{|\tilde{J}'(0)|}$.

$J(t) = (d\exp_p)_{t\gamma'(0)}(t \cdot J'(0))$

$\tilde{J}(t) = (d\exp_{\tilde{p}})_{t\tilde{\gamma}'(0)}(t \cdot \tilde{J}'(0))$

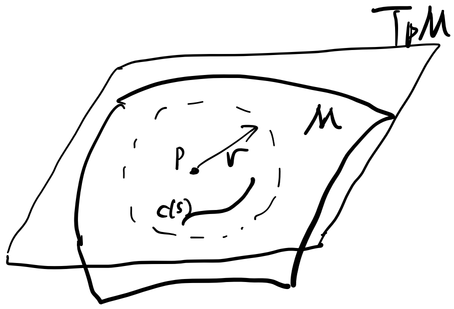


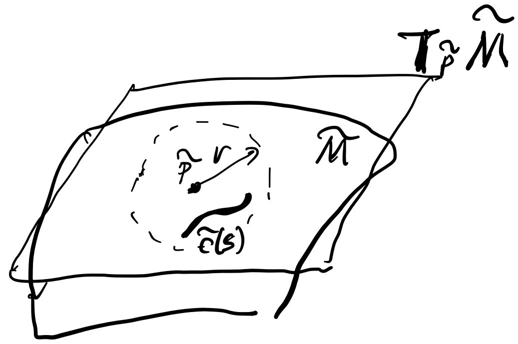
- $\tilde{\gamma}$ has no conjugate points
- $\tilde{K}(\tilde{V}, \tilde{\gamma}'(t)) \geq K(V, \gamma'(t))$. $\forall t \in [0,a]$

$\Rightarrow \quad |\tilde{J}(t)| \leq |J(t)|$.

Application:

Prop:





$\exp_p|B_r(0)$ is a diffeom. $\qquad$ $\exp_{\tilde{p}}|\tilde{B}_r(0)$ is non-singular.

$B_r(0) \subset T_pM$

$W = \exp_p(B_r(0))$ $\qquad$ $\tilde{W} = \exp_{\tilde{p}}(\tilde{B}_r(0))$

$i: T_pM \to T_{\tilde{p}}M$ isometry. $\qquad$ $c: [0,a] \to W$.

$$\tilde{c}(s) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(c(s)) : [0,a] \to \tilde{W}$$

Assume $K_{\tilde{M}} \geq K_M$ at any points.

$$\Longrightarrow \quad l(c) \geq l(\tilde{c}).$$

Special case:

M is simply connected $\underline{K(\sigma) \le 0}$ $\forall \sigma \subset T_pM$

$\xRightarrow{\text{Hadamard Thm}}$ $M \cong \mathbb{R}^n$.

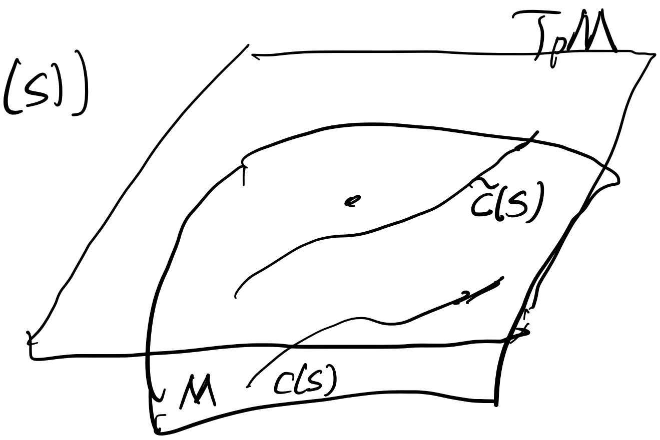
$\exp_p : T_pM \to M$ is a diffeomorphism.

$$T_pM = \widetilde{M}$$

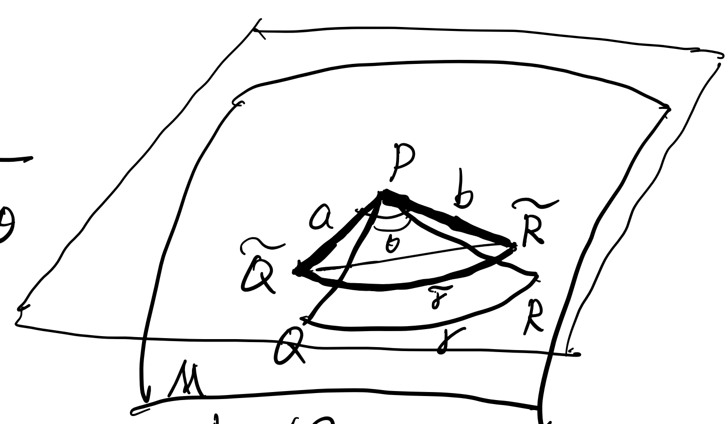
$K_M \le K_{\widetilde{M}} = 0$. $\quad c : [0, a] \to M$

$\Rightarrow \tilde{c}(s) = \exp_p^{-1}(c(s))$

$\Rightarrow l(\tilde{c}(s)) \le l(c(s))$

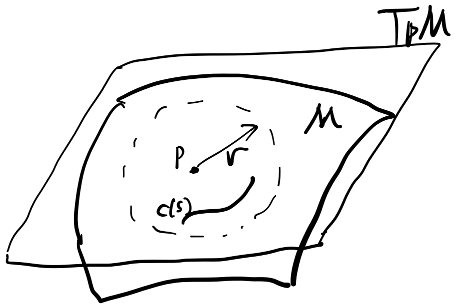


$$l(\tilde{\gamma}) \le l(\gamma) = c$$
$$\vee\!/$$
$$l(\overline{\tilde{Q}\tilde{R}}) = \sqrt{a^2+b^2-2ab\cos\theta}$$



$\Rightarrow c^2 \ge a^2+b^2-2ab\cos\theta \Leftarrow K_M \le 0$

Proof of Prop:



$\exp_p|B_r(0)$ is a diffeom. $\qquad$ $\exp_{\tilde p}|\tilde B_r(0)$ is non-singular.

$B_r(0) \subset T_pM$

$W = \exp_p(B_r(0))$ $\qquad$ $\tilde W = \exp_{\tilde p}(\tilde B_r(0))$

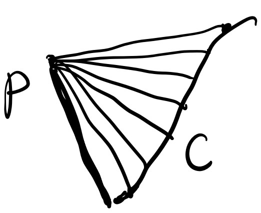
$i : T_pM \to T_{\tilde p}M$ isometry. $\qquad$ $c : [0,a] \to W$.

$\tilde c(s) = \exp_{\tilde p} \circ i \circ \exp_p^{-1}(c(s)) : [0,a] \to \tilde W$

Assume $\tilde K_{\tilde M} \geq K_M$ at any points.

$\Longrightarrow \quad l(c) \geq l(\tilde c)$.

Pf:



$c(s) \subseteq \exp_p(B_r(0))$

$c(s) = \exp_p(v(s))$, $\quad v(s) \in B_r(0)$

$f(t,s) = \exp_p(t \cdot v(s))$. $\quad f(1,s) = c(s)$

$\tilde f(t,s) = \exp_{\tilde p}(t \cdot i(v(s))) = \exp_{\tilde p}(t \cdot \tilde v(s))$

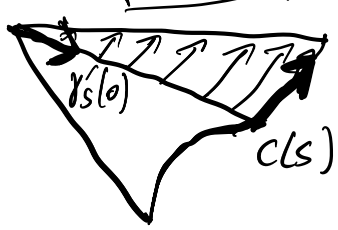
$i: T_pM \to T_{\tilde p}\tilde M$.

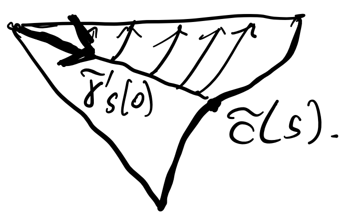
$$\tilde f(1,s) = \exp_{\tilde p}\left(i\cdot \exp_p^{-1}(c(s))\right) = \tilde c(s).$$

$$c'(s) = \frac{\partial}{\partial s} f(t,s)\Big|_{t=1} = (d\exp_p)_{v(s)}(v'(s)) = \boxed{J_s(1)}$$

$$J_s(t) = \frac{\partial}{\partial s} f(t,s) = \underline{(d\exp_p)_{t\cdot v(s)}(t\cdot v'(s))}$$

$$\tilde c'(s) = \boxed{\tilde J_s(1)}, \quad \tilde J_s(t) = \frac{\partial}{\partial s}\tilde f(t,s)$$





$J_s(0) = 0 = \tilde J_s(0)$.

$$\underline{J_s'(0) = v'(s)} \qquad i(v(s)) = \tilde v(s)$$

$$\underline{\tilde J_s'(0) = \tilde v'(s)} = i(v'(s))$$

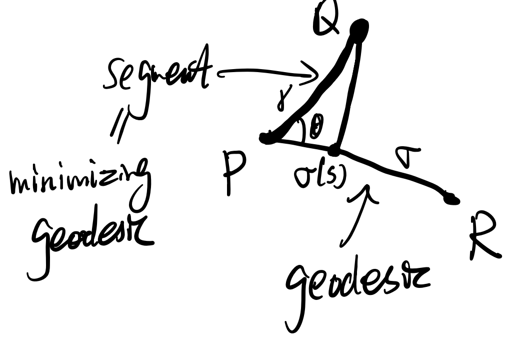
$$\langle J_s'(0), \gamma_s'(0)\rangle = \langle i(J_s'(0)), \underline{i(\gamma_s'(0))}\rangle = \langle \tilde J_s'(0), \underline{\tilde\gamma_s'(0)}\rangle$$

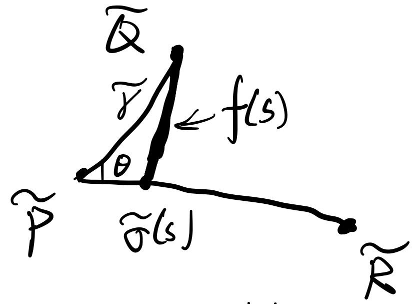
$$\Rightarrow \underset{|c'(s)|}{\overset{}{|J_s(1)|}} \geq \underset{|\tilde c'(s)|}{|\tilde J_s(1)|} \Rightarrow \int_0^a |c'(s)|\,ds = l(c)$$

$$\geq \qquad\qquad \geq$$

$$\int_0^a |\tilde c'(s)|\,ds = l(\tilde c).$$

Toponogov Comparison Thm: M $\quad \underline{K_M \geq \lambda}$





in M_λ
$\parallel$
space form with constant curvature λ.

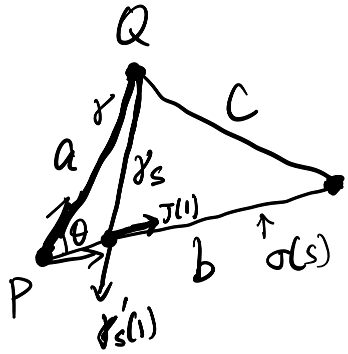
$$\Rightarrow \quad d(Q, \sigma(t)) \leq d(\tilde{Q}, \tilde{\sigma}(t)).$$

If $\underline{\lambda = 0}$, $\quad d(Q, \sigma(t)) \leq d(\tilde{Q}, \tilde{\sigma}(t)) \Leftarrow \underline{\text{cosine law}}.$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\theta.$$

We will first derive cosine law for space forms with constant curvatures.

M_λ = space form with constant curvature λ.



$$\underbrace{d(Q, \sigma(s))}_{\displaystyle \int_0^1 |\gamma_s'(t)|\,dt} = r(s) = \ell(\gamma_s)$$

$\cos\theta = \langle \gamma'(0), \sigma'(0) \rangle$ on T_pM.

$$E(\gamma_s) = \int_0^1 |\gamma_s'(t)|^2 dt = \left(\int_0^1 |\gamma_s'(t)|\,dt\right)^2 = r(s)^2$$

$$\Rightarrow \frac{d}{ds} E(\gamma_s) = 2 \cdot r(s) \cdot \frac{d}{ds} r(s)$$

$$\frac{1}{2}\frac{d}{ds} E(\gamma_s) = \frac{1}{2}\frac{d}{ds}\int_0^1 |\gamma'(t)|^2 dt = \int_0^1 \langle \underbrace{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}}_{\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}}, \frac{\partial}{\partial t} \rangle dt = \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle (1)$$

$$\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\rangle = \frac{\partial}{\partial t}\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\rangle - \langle \frac{\partial}{\partial s}, \underbrace{\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}}_{0} \rangle$$

$$= \langle J(1), \gamma'(1) \rangle$$

$$\Rightarrow \boxed{\frac{d}{ds} r(s) = \frac{1}{2r(s)}\frac{d}{ds} E(\gamma_s) = \frac{1}{r(s)} \langle J(1), \gamma'(1) \rangle.} \quad (1)$$

tangent component

$$\frac{1}{2}\frac{d^2}{ds^2} E(\gamma_s) = \int_0^1 \left(\underbrace{\langle J', J' \rangle - \langle R(\gamma', J)J, \gamma' \rangle}_{\frac{d}{dt}\langle J', J\rangle} \right) dt$$

$$= \langle J'(1), J(1) \rangle$$

For normal Jacobi field

$a = r(s)$

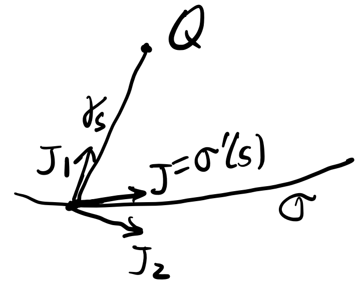
$$J_2(t) = \begin{cases} \dfrac{\sin(a\cdot t\sqrt{\lambda})}{\sqrt{\lambda}}\, w(t) & \lambda > 0 \\[2mm] a\cdot t\cdot w(t) & \lambda = 0 \\[2mm] \dfrac{\sinh(a t\sqrt{-\lambda})}{\sqrt{-\lambda}}\, w(t) & \lambda < 0 \end{cases} = \underline{sn_\lambda(at)}\cdot w(t)$$

parallel v.f. along geod.

$$J_2'(t) = a\cdot sn_\lambda'(at)\cdot w(t) = a\cdot \frac{sn_\lambda'(at)}{sn_\lambda(at)}\, J_2(t)$$

$$\langle J_2'(1), J_2(1)\rangle = a\cdot\left(\frac{sn_\lambda'(a)}{sn_\lambda(a)}\right)|J_2(1)|^2 = r(s)\cdot\left(\frac{sn_\lambda'(r(s))}{sn_\lambda(r(s))}\right)|J_2(1)|^2$$

General Jacobi field



$$J = J_1 + J_2$$

$$J(t) = \frac{\partial}{\partial s}\exp_Q(t\cdot\gamma_s'(0)). \quad J(1) = \frac{\partial}{\partial s}\exp_Q(\gamma_s'(0)) = \frac{\partial}{\partial s}\sigma(s) = \sigma'(s).$$

$$J = J_1 + J_2 = t\cdot\frac{\langle J,\gamma'\rangle}{r(s)}\frac{\gamma'}{r(s)} + sn_\lambda(t)\cdot w(t) \quad (\text{tangent} + \text{normal})$$

$$J'(1) = J_1(1) + J_2'(1)$$

$$\Rightarrow \langle J'(1), J(1)\rangle = |J_1(1)|^2 + \langle J_2'(1), J_2(1)\rangle$$

$$= |J_1(1)|^2 + r(s)\cdot\left(\frac{sn_\lambda'(r(s))}{sn_\lambda(r(s))}\right)\cdot|J_2(1)|^2.$$

$$\frac{1}{2}\frac{d^2}{ds^2}E(s) = \frac{d^2}{ds^2}\frac{1}{2}r(s)^2 = \frac{d}{ds}(r\cdot r') = r'^2 + r\cdot r''$$

$$\| \textcircled{1}$$

$$\left(\frac{1}{r(s)^2}\langle J(1), \gamma'(1)\rangle\right)^2 + r(s)\cdot r''$$

$$\|$$

$$|J_1|^2 + r(s)\cdot r''$$

$$\|$$

$$|J_1|^2 + r(s)\left(\frac{sn_\lambda'(r)}{sn_\lambda(r)}\right)|J_2|^2$$

$$\Rightarrow \quad r'' = \frac{d}{ds^2}r(s) = \left(\frac{sn_\lambda'(r)}{sn_\lambda(r)}\right)\cdot|J_2(1)|^2 \qquad \textcircled{2}$$

$$r' = \frac{d}{ds}r(s) = \frac{1}{r(s)}\underline{\langle J(1), \gamma'(1)\rangle} = \langle J, \frac{\gamma'}{r}\rangle \qquad \textcircled{1}$$

Find $\varphi(s) = h(r(s))$. s.t. $\varphi(s)$ satisfies a 2nd order ODE

$$\varphi' = h_r\cdot r', \quad \varphi'' = h_{rr}\cdot r'^2 + h_r\cdot r''$$

$$\xRightarrow{\textcircled{1}+\textcircled{2}} \varphi'' = h_{rr}\cdot|J_1|^2 + h_r\cdot\frac{sn_\lambda'(r)}{sn_\lambda(r)}|J_2|^2$$

Need $h_{rr} = h_r\cdot\frac{sn_\lambda'(r)}{sn_\lambda(r)} = b(r)$

$$\Rightarrow \varphi'' = b(r(s))\left(|J_1|^2 + |J_2|^2\right) = b(r(s))\cdot|J(1)|^2 = b(r(s)).$$

$$|J(1)|^2 = (\sigma'(s))^2$$

Case 0: $\lambda = 0$. Choose $h = r^2$, $sn_0(t) = t$

$\Rightarrow h_{rr} = 2$, $h_r \cdot \frac{sn_0'(r)}{sn_0(r)} = 2r \cdot \frac{1}{r} = 2$. ✓

Cases when $\lambda \neq 0$. ?