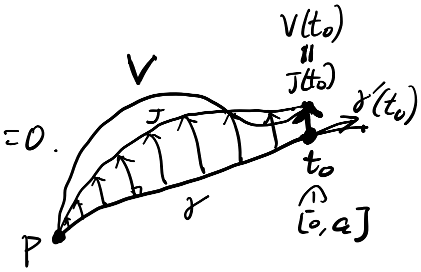
Index Lemma: $\gamma: [0, a] \to M$ geodesic without conjugate points.

J: Jacobi field. $J(0)=0$. $\langle J, \gamma' \rangle = 0$

V: piecewise differentiable v.f. $V(0)=0$. $\langle V, \gamma' \rangle = 0$.

$V(t_0) = J(t_0) \in \gamma'(t_0)^{\perp} \subset T_{\gamma(t_0)}M$



Then: $\underline{I_{t_0}(J,J) \leq I_{t_0}(V,V)} \longleftarrow \forall V$

$J(t) = (d\exp_{\gamma(0)})_{t_0\gamma'(t_0)}(t_0 w)$

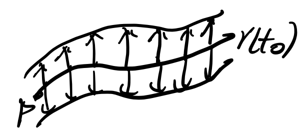
"=" holds iff $V=J$ on $[0, t_0]$

No conjugate points along γ $\Downarrow$

$(d\exp_{\gamma(t_0)})_{t\gamma'(t_0)}: T_pM \to T_{\gamma(t_0)}M$ is invertible

Index Form:

$$I_{t_0}(V,V) = \int_0^{t_0} \left(\langle V', V' \rangle - \langle R(\gamma', V)V, \gamma' \rangle \right) dt$$



$$\frac{1}{2}\frac{d^2}{ds^2} E(\gamma_s) = I_{t_0}(V,V) + \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \gamma' \right\rangle \Big|_0^{t_0}$$

$\gamma_0(t) = \gamma(t)$ is a geodesic

$\Updownarrow$

γ is a critical point of Energy functional

$\Updownarrow$

Hessian of $f(x)$ at a critical point $0 \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}$

$$f(x) = f(0) + \underbrace{\sum_i \frac{\partial f}{\partial x_i}(0) x_i}_{= 0} + \underbrace{\frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \cdot x_i \cdot x_j}_{= \frac{1}{2} \cdot \mathrm{Hess} f(x,x)} + o(|x|^2)$$

|| coordinate change. $\exists \lambda_i > 0$, $i=1, \cdots, n$.

$\lambda_1 x_1^2 + \cdots + \lambda_p x_p^2 - (\lambda_{p+1} x_{p+1}^2 + \cdots + \lambda_{n} x_n^2)$, $n = p+q$

Bilinear form of (p, q) type

$\uparrow$ index of $\mathrm{Hess}(f)$.

Pf: $\gamma(t)$ has no conjugate points. $t\in[0,a]$.

$\Rightarrow$ $J_i(t)$, $i=1,\cdots,n-1$. Jacobi fields along $\gamma(t)$

$\underset{\underset{\displaystyle \underline{(d\exp_p)_{t\gamma'(0)}(tw_i)}}{\parallel}}{}$ $\qquad \underline{\{w_i\} \text{ basis for } \gamma'(0)^\perp} \subset T_{\gamma(0)}M$, $\gamma(0)=p$. $\quad$ $J_j'(0)$

form basis of $\gamma'(t)^\perp \subset T_{\gamma(t)}M$.

$\Rightarrow$ $V(t)=\sum_{i=1}^{n-1} f_i(t)\cdot J_i(t) \quad \in \gamma'(t)^\perp$.

$V(t_0)=\sum_{i=1}^{n-1} \underline{f_i(t_0)}\cdot J_i(t_0) = J(t_0)$.

Claim: $\langle V',V'\rangle - \langle R(\gamma',V)V,\gamma'\rangle = \left|\sum_i f_i' J_i\right|^2 + \frac{d}{dt}\langle \sum_i f_i J_i, \sum_j f_j J_j'\rangle$

$\Rightarrow I_{t_0}(V,V) = \underbrace{\int_0^{t_0} \left|\sum f_i' J_i\right|^2 dt}_{\geq 0} + \langle \underbrace{\sum_i f_i J_i}_{V}, \sum_j f_j J_j' \rangle \Big|_0^{t_0}$

$\qquad \parallel$

$\underline{\langle V(t_0), \sum_j f_j(t_0)\cdot J_j'(t_0)\rangle}$

$I_{t_0}(J,J) =$

$J = \sum_i a_i(t)\cdot \underline{J_i}$. $\Rightarrow$ $a_i(t)\equiv a_i$ constant.

$\nearrow$

$0 = J'' - \langle R(\gamma',V)V,\gamma'\rangle$

$\Rightarrow I_{t_0}(J,J) = \langle \underset{\underset{\displaystyle V(t_0)}{\parallel}}{J(t_0)}, \sum_j \underset{\underset{\displaystyle a_j=f_j(t_0)}{\parallel}}{\underline{a_j(t_0)}}\cdot J_j'(t_0)\rangle = \underline{\langle V(t_0), \sum_j f_j(t_0) J_j'(t_0)\rangle}$

$\Rightarrow I_{t_0}(V,V) - I_{t_0}(J,J) = \int_0^{t_0} \left|\sum_j f_j' J_j\right|^2 dt \geq 0$. $\quad$ "=" 时 $f_j'\equiv 0$

$\Updownarrow$

$V(t)\equiv J(t),\ t\in[0,t_0]$

Claim: $\langle V',V'\rangle - \langle R(\gamma',V)V,\gamma'\rangle = \left|\sum_i f_i' J_i\right|^2 + \frac{d}{dt}\langle \sum_i f_i J_i, \sum_j f_j J_j'\rangle$

$V = \sum_i f_i J_i$, $\underline{V' = \sum_i (f_i' J_i + f_i J_i')}$

$$\langle R(\gamma',V)V,\gamma'\rangle = -\langle R(\gamma',V)\gamma',V\rangle = -\langle R(\gamma',\sum_i f_i J_i)\gamma',V\rangle$$

$$= -\sum_i f_i \langle \underbrace{R(\gamma',J_i)\gamma'}_{=J_i''},V\rangle$$

$$= -\langle \sum_i f_i J_i'', \sum_i f_i J_i\rangle$$

$$\langle V',V'\rangle - \langle R(\gamma',V)V,\gamma'\rangle = \langle \sum_i (f_i' J_i + f_i J_i'), \sum_j (f_j' J_j + f_j J_j')\rangle$$
$$+ \langle \sum_i f_i J_i'', \sum_i f_i J_i\rangle \rightsquigarrow \text{Claim.}$$

$$I_{t_0}(V,V) = \int_0^{t_0} (\underbrace{\langle V',V'\rangle}_{=\left(\frac{d}{dt}\langle V',V\rangle\right) - \langle V'',V\rangle} - \langle R(\gamma',V)V,\gamma'\rangle)\,dt$$

$$\frac{d}{dt}\langle V',V\rangle = \langle V'',V\rangle + \langle V',V'\rangle$$

V differentiable

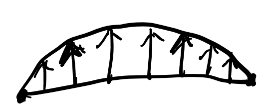
$$= \underline{\langle V',V\rangle\Big|_0^{t_0}} - \int_0^{t_0}(\langle V'',V\rangle + \langle R(\gamma',V)V,\gamma'\rangle)\,dt.$$

$$\underline{I_{t_0}(J,J)} = \langle J',J\rangle\Big|_0^{t_0} - \int_0^{t_0}(\underline{\langle J'',J\rangle - \langle R(\gamma',J)\gamma, J\rangle})\,dt$$

J Jacobi field $= \langle J',J\rangle(t_0) - \underline{\langle J',J\rangle(0)}$

$\underline{J(0)=0}$ $= \langle J',J\rangle(t_0)$

$J(t_0)=0$ $= 0.$

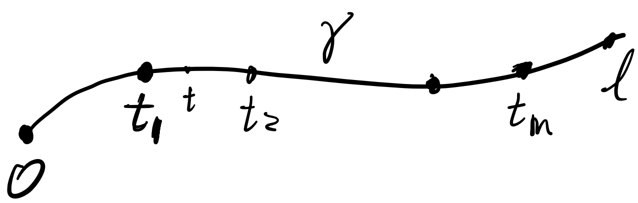


$$I_{t_0}(V,W)=\int_0^{t_0}\left(\langle V',W'\rangle - \langle R(\gamma',V)W,\gamma'\rangle\right)dt$$

$$\mathcal{V}(0,t_0)=\left\{\begin{array}{l}\text{normal vector fields } V \text{ along } \overset{\text{geodesic}}{\gamma:[0,t_0]\to M},\\ V \text{ is piecewise differentiable, } V(0)=V(t_0)=0\end{array}\right\}.$$

$V\in \mathrm{null}(I_{t_0})$ if $I_{t_0}(V,W)=0 \quad \forall\, W\in \mathcal{V}(0,t_0)$.

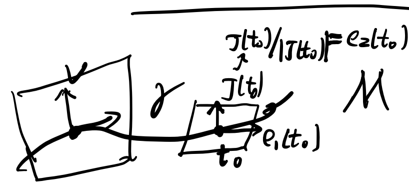
Prop: $\mathrm{null}(I_{t_0}) = \{\text{Jacobi fields } J\in\mathcal{V}(0,t_0)\}$.

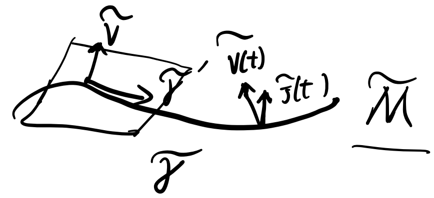


Rauch comparison: Geodesics $\gamma:[0,a]\to M$, $\tilde\gamma:[0,a]\to\tilde M$. J, $\tilde J$: Jacobi fields

- $|\gamma'(t)|=|\tilde\gamma'(t)|$
- $J(0)=\tilde J(0)=0$. $\underset{\underset{|J'(0)|\cdot|\gamma'(0)|\cdot\cos\theta}{\|}}{\langle J'(0),\gamma'(0)\rangle}=\langle\tilde J'(0),\tilde\gamma'(0)\rangle$. $|J'(0)|=|\tilde J'(0)|$.
- $\tilde\gamma$ has no conjugate points.
- $\tilde K(\tilde v,\tilde\gamma'(t))\geq K(x,\gamma'(t))$.

sectional curvature





$\Rightarrow$ $\underline{|\tilde J(t)| \le |J(t)| \quad \forall t \in [0,a].}$

If for some $t_0 \in [0,a)$ $|\tilde J(t_0)| = |J(t_0)| \Rightarrow$ $\tilde K(\tilde J(t), \tilde\gamma'(t)) = K(J(t), \gamma'(t))$ $\forall t \in [0, t_0]$.

Pf: For simplicity, assume $J \perp \gamma'$ and $\tilde J \perp \tilde\gamma'$.

($J = J^{\perp} + J^{p}$ in general)

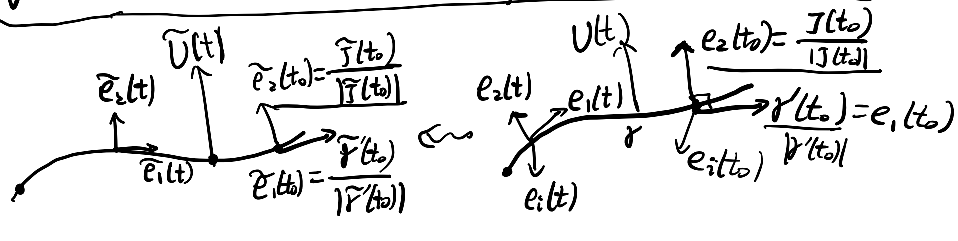
$$v(t) = \underline{|J(t)|^2} \ngeq \tilde v(t) = \underline{|\tilde J(t)|^2}.$$

$$f(t) = \frac{v(t)}{\tilde v(t)}, \qquad f(0) = \lim_{t\to 0} f(t) = \frac{|J'(0)|^2}{|\tilde J'(0)|} = 1.$$

$$f' = \frac{v'\cdot\tilde v - v\cdot\tilde v'}{\tilde v(t)^2} = \left(\frac{v'}{v} - \frac{\tilde v'}{\tilde v}\right)\cdot\frac{v\cdot\tilde v}{\tilde v^2} = \boxed{\left(\frac{v'}{v} - \frac{\tilde v'}{\tilde v}\right)}\cdot\left(\frac{v}{\tilde v}\right) \ge 0.$$

$$\frac{v'}{v}\Big|_{t_0} = \frac{\frac{d}{dt}\langle J, J\rangle}{|J|^2} = \frac{2\cdot\langle J', J\rangle}{|J|^2}(t_0) = 2\cdot\frac{I_{t_0}(J,J)}{|J(t_0)|^2} \qquad \boxed{\begin{matrix} I_t(J,J) \\ \shortparallel \\ \langle J'(t), J(t)\rangle \end{matrix}}$$

$$\frac{\tilde v'}{\tilde v} = 2\cdot I_{t_0}\left(\frac{\tilde J}{|\tilde J(t_0)|}, \frac{\tilde J}{|\tilde J(t_0)|}\right) \nleq 2\, I_{t_0}\left(\frac{J}{|J(t_0)|}, \underbrace{\frac{J}{|J(t_0)|}}_{e_2(t)}\right)$$



$$U(t) = \frac{J(t)}{|J(t_0)|} = \sum_{i=1}^{n} \underline{a_i(t)} \cdot \underline{e_i(t)} \quad \left(\Rightarrow \underline{U''(t) = R(\gamma', U)\gamma'} \right)$$

$$\tilde{V}(t) = \sum_{i=1}^{n} \underline{a_i(t)} \cdot \tilde{e}_i(t).$$ vector field along $\tilde{\gamma}$

$$\underbrace{I_{t_0}(U(t), U(t))}_{\text{VI}} = \int_0^{t_0} \big(\langle U', \underset{\text{II}}{U'} \rangle \underbrace{- \langle R(\gamma', U)U, \gamma' \rangle}_{\text{VI}} \big) dt$$

$$I_{t_0}(\tilde{V}(t), \tilde{V}(t)) = \int_0^{t_0} \big(\langle \tilde{V}', \tilde{V}' \rangle \underline{- \langle \tilde{R}(\tilde{\gamma}', \tilde{V})\tilde{V}, \tilde{\gamma}' \rangle} \big) dt$$

$$|U(t)|^2 = |\tilde{V}(t)|^2 = \sum_i |a_i(t)|^2 \qquad \boxed{\tilde{K}(\tilde{\gamma}', \tilde{V}) \geq K(\gamma', V)}$$

$$K(\gamma', V) = \frac{\langle R(\gamma', U)V, \gamma' \rangle}{|\gamma'| \cdot |V|}$$

$$U'(t) = \sum_i a_i'(t) \cdot e_i(t) \Rightarrow |U'(t)|^2 = |\tilde{V}'(t)|^2$$

$$\tilde{V}'(t) = \sum_i a_i'(t) \cdot \tilde{e}_i(t)$$

$$U(t_0) = \frac{J(t_0)}{|J(t_0)|} = e_2(t_0) \Rightarrow \underline{\tilde{V}(t_0) = \tilde{e}_2(t_0) = \frac{\tilde{J}(t_0)}{|\tilde{J}(t_0)|}}.$$

Index Lemma

$$\Rightarrow \underline{I_{t_0}(\tilde{V}(t), \tilde{V}(t)) \geq I_{t_0}\left(\frac{\tilde{J}(t)}{|\tilde{J}(t_0)|}, \frac{\tilde{J}(t)}{|\tilde{J}(t_0)|} \right)}$$

$$\Rightarrow I_{t_0}\left(\frac{J(t)}{|J(t_0)|}, \frac{J(t)}{|J(t_0)|} \right) \geq I_{t_0}\left(\frac{\tilde{J}(t)}{|\tilde{J}(t_0)|}, \frac{\tilde{J}(t)}{|\tilde{J}(t_0)|} \right).$$

"=" holds $\Leftrightarrow$ $\tilde{V}(t) \equiv \frac{\tilde{J}(t)}{|\tilde{J}(t_0)|}$ and $\langle R(\gamma', U(t))U(t), \gamma' \rangle$

$$\shortparallel$$

$$\langle R(\tilde{\gamma}', \tilde{U}(t))\tilde{U}(t), \tilde{\gamma}' \rangle. \quad \blacksquare$$