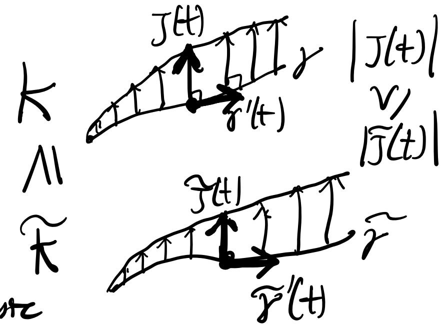
Rauch comparison Thm

(M, g) $\quad \gamma : [0, l] \to M$ geodesic $\quad |\gamma'(t)| \equiv 1$. (normalized)

J : Jacobi field along γ. $\quad J(0) = 0$. $\quad |J'(0)| = 1$, $\langle J'(0), \gamma'(0) \rangle = 0$.

$$J = \left(d\mathrm{Exp}_{\gamma(t)}\right)_{t\gamma'(0)}\left(t \cdot J'(0)\right) = \frac{\partial}{\partial s} \mathrm{Exp}_p\left(t(\gamma'(0) + s \cdot J'(0))\right)\Big|_{s=0}.$$



$(\tilde{M}, \tilde{g})$. $\quad \tilde{\gamma} : [0, l] \to \tilde{M}$ normalized geodesic

$\tilde{J}$: Jacobi field, $\tilde{J}(0) = 0$, $|\tilde{J}'(0)| = 1$. $\langle \tilde{J}'(0), \tilde{\gamma}'(0) \rangle = 0$.

$$\frac{d}{dt}\langle J(t), \gamma'(t) \rangle = \langle J', \gamma' \rangle + \langle J, \underset{0}{\gamma''} \rangle = \langle J', \gamma' \rangle$$

$$\frac{d^2}{dt^2}\langle J(t), \gamma'(t) \rangle = \frac{d}{dt}\langle J', \gamma' \rangle = \langle \underset{\langle R(\gamma', J)\gamma', \gamma' \rangle = 0}{J''}, \gamma' \rangle + \langle J', \underset{0}{\gamma''} \rangle = 0.$$

$$\langle J(t), \gamma'(t) \rangle = \langle J(0), \gamma'(0) \rangle + \langle J'(0), \gamma'(0) \rangle \cdot t.$$

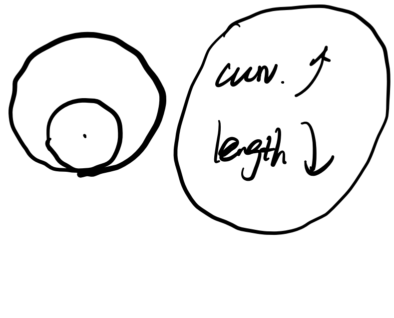
$$J(0) = 0, \ \langle J'(0), \gamma'(0) \rangle = 0 \implies \langle J(t), \gamma'(t) \rangle \equiv 0, \ t \in [0, l]$$

$$\Updownarrow$$

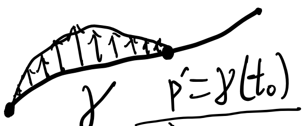
Thm (Rauch) $\tilde{\gamma} : [0, l]$ does not have conj. points. $\quad J(t) \perp \gamma'(t)$

If $K(\tilde{\gamma}'(t), \tilde{J}(t)) \geq K(\gamma'(t), J(t))$, $t \in [0, l]$

Then $|\tilde{J}(t)| \leq |J(t)|$.



Prop: $\underbrace{0 < L \le K \le \overset{\text{constant}}{H}}_{K\, =\, \text{sectional curv. of } (M,g)}$. $\gamma : [0,+\infty) \to M$ geodesic



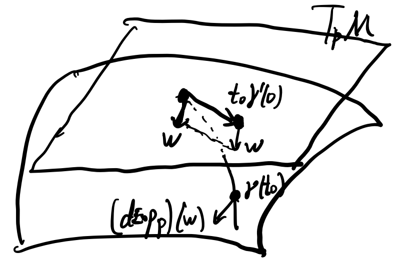
1st. conjugate points for p along γ $\longleftrightarrow$ $\exists$ a Jacobi field $J \not\equiv 0$. s.t. $J(0) = 0$, $J(t_0) = 0$.

$\Updownarrow$

$(d\exp_p)_{t_0\gamma'(0)} : \underset{\cong\, T_{t_0\gamma'(0)}(T_pM)}{T_pM} \to \underline{T_{\gamma(t_0)}M}$ is singular

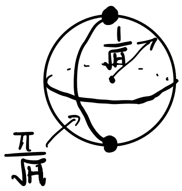
Then: $\frac{\pi}{\sqrt{H}} \le L(p \xrightarrow{\gamma} p') \le \frac{\pi}{\sqrt{L}}$.

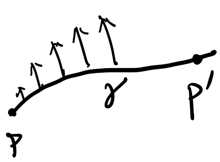
$J(t) = (d\exp_p)_{t\gamma'(0)}(t J'(0))$

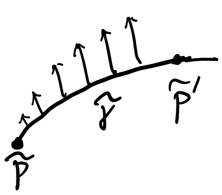


Pf: $K \le H \overset{\text{Rauch}}{\Rightarrow} \underline{|J(t)| \ge |\tilde{J}(t)|}$

$\Updownarrow$

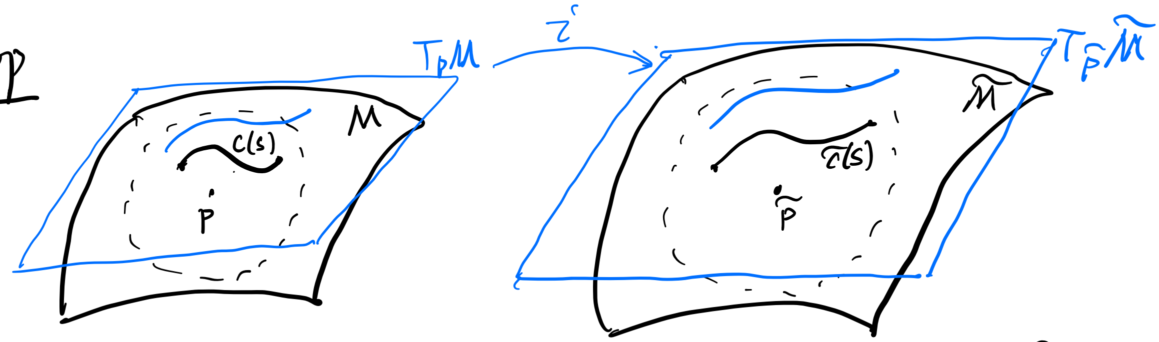
$S^n_H =$ 





$L(p \xrightarrow{\gamma} p') \ge L(\tilde{p} \xrightarrow{\tilde{\gamma}} \tilde{p}') = \frac{\pi}{\sqrt{H}}$.

Prop



Assume $\exp_p : \underset{\displaystyle T_pM}{\overset{\cap}{B_r(0)}} \longrightarrow M$ is a diffeomorphism.

$\exp_{\tilde p} : \underset{\displaystyle T_{\tilde p}\tilde M}{\overset{\cap}{\tilde B_r(0)}} \longrightarrow \tilde M$ is nonsingular (i.e. differential $d\exp_p$ is invertible)

Fix a linear isometry $i : T_pM \longrightarrow T_{\tilde p}\tilde M$.

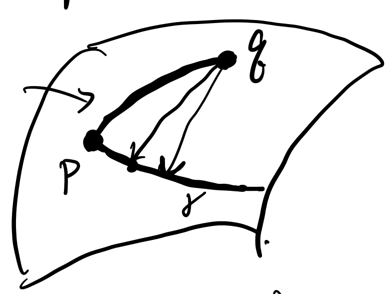
$\tilde c(s) = \exp_{\tilde p} \circ i \circ \exp_p^{-1}(c(s))$.

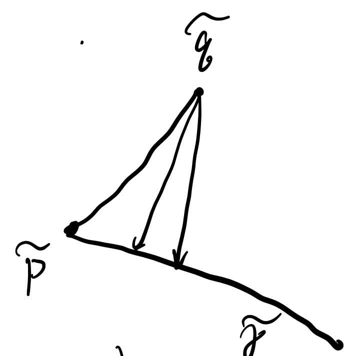
If $K_{\tilde M} \geq K_M$, then $\ell(\tilde c) \leq \ell(c)$.

$K_{\tilde M}(\tilde\sigma) \geq K_M(\sigma)$

Toponogov Comparison: $K \geq H$.

Hinge





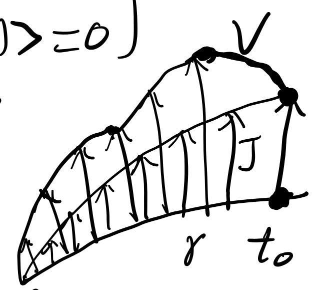
$$d(q, \gamma(t)) \leq d(\tilde q, \tilde\gamma(t)).$$

• **Index lemma** : $\gamma : [0, a] \to M$ geodesic without conjugate points to $\gamma(0)$

J : Jacobi field. $J(0) = 0$, $\langle J, \gamma' \rangle = 0$

$\Updownarrow$

$\left(\langle J'(0), \gamma'(0) \rangle = 0 \right)$



V : piecewise differentiable vector field along γ

satisfying $V(0) = 0$, $\langle V, \gamma' \rangle = 0$.

Fix $t_0 \in [0, a]$. If $J(t_0) = V(t_0)$,

Then $\boxed{I_{t_0}(J, J) \le I_{t_0}(V, V)}$ and

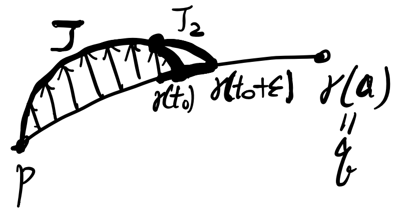
"=" holds iff $J \equiv V$ on $[0, t_0]$.

$$\underline{I_{t_0}(V, V)} = \int_0^{t_0} \left(\langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \right) dt.$$

$$\frac{1}{2} \frac{d^2}{ds^2} E(c_s) = \boxed{I_{t_0}(V, V)} + \underline{\left\langle \nabla_{\frac{\partial}{\partial s}} V(s)\Big|_{s=0}, \gamma'(t) \right\rangle} \Big|_0^{t_0}$$

(Jacobi)

Application of Index Lemma: If $\gamma : [0,a] \xrightarrow{\text{geodesic}} M$ has a conjugate point $\gamma(t_0)$ to $\gamma(0)$ for $t_0 \in [0,a)$, then γ is not length minimizing.



$$\text{vector field } \underset{\text{(Broken Jacobi)}}{V(t)} = \begin{cases} \overset{J(t)}{\overset{\|}{J_1(t)}} & t \le t_0 - \varepsilon \\ J_2(t) & t_0-\varepsilon \le t \le t_0+\varepsilon \end{cases}$$

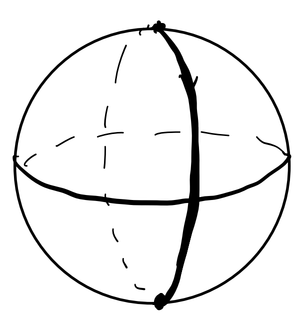
$J_2(t)$ Jacobi field: $\underline{J_2(t_0+\varepsilon) = 0, \quad J_2(t_0-\varepsilon) = J_1(t_0-\varepsilon)}$

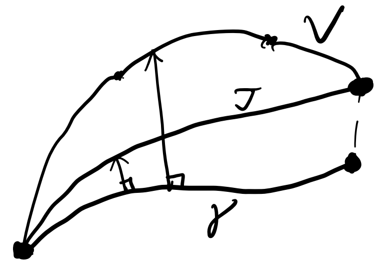
$$\frac{1}{2}\frac{d^2}{ds^2} E(\underset{\underset{V(t)}{\updownarrow}}{C_s}) = \underline{I_{t_0+\varepsilon}(V,V)} = \int_0^{t_0-\varepsilon} B(V,V) + \underbrace{\int_{t_0-\varepsilon}^{t_0+\varepsilon} B(V,V)}_{\wedge \text{ Index Lemma}}$$

$$0 = \int_0^{t_0} B(J,J)\,dt = \int_0^{t_0-\varepsilon} B(W,W) + \underline{\int_{t_0-\varepsilon}^{t_0+\varepsilon} B(W,W)\,dt}$$

$$\|$$

$$\int_0^{t_0} \left(\langle J', J' \rangle - \langle R(\gamma', J)\gamma', J \rangle \right) dt$$





Proof of Index Lemma

$\gamma(t)$ has no conjugate points for $t \in [0, a]$

$\rightsquigarrow$ Find $J_i(t)$, $i=1,\dots,n-1$, $(n-1)$-Jacobi fields.

s.t. $\{J_i(t),\ i=1,\dots,n-1\}$ form a basis for $\gamma'(t)^{\perp} \subset T_{\gamma(t)}M$, $t \neq 0$.

Choose a basis $\{w_1,\dots,w_{n-1}\}$ for $\gamma'(0)^{\perp}$

$$\boxed{J_i(t) = (d\exp_p)_{t\gamma'(0)}(t\cdot w_i),\ i=1,\dots,n.}$$

$V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t)$, $f_i(0)=0$. f_i: pcws. diff.

$V' = \sum_i f_i' J_i + f_i J_i'$

Claim $B(V,V) = \langle V', V' \rangle - \langle R(\gamma', V)V, \gamma' \rangle$

$$= \underline{\left|\sum_i f_i' J_i\right|^2 + \frac{d}{dt}\left\langle \sum_i f_i J_i, \sum_j f_j J_j' \right\rangle}.$$

$$I_{t_0}(V,V) = \int_0^{t_0} B(V,V)\,dt = \left\langle \overset{V}{\sum_i f_i J_i}, \sum_j f_j J_j' \right\rangle\Big|_0^{t_0} + \underbrace{\int_0^{t_0}\left|\sum_i f_i' J_i\right|^2 dt}_{\geq 0}$$

$$\geq$$

$$I_{t_0}(J,J) = \int_0^{t_0} B(J,J)\,dt = \underline{\langle J, J' \rangle\big|_0^{t_0}}$$

$$\frac{d^2}{dt^2}\langle J, \gamma' \rangle = \underset{\underset{R(\gamma', J)\gamma'}{\shortparallel}}{\langle J'', \gamma' \rangle} = 0 \ . \quad \langle J, \gamma' \rangle = t \cdot C.$$

$$C = \underset{\underset{w}{\shortparallel}}{\langle J'(0), \gamma' \rangle}$$

$$V = \sum_i f_i J_i \ . \quad f_j(t_0) = a_i \ .$$

$$J = \sum_i a_i J_i \Rightarrow J' = \sum_i a_i \cdot J_i' \ . \quad \sum_j f_j(t_0) J_j' = \sum_j a_j J_j'(t_0) = J'(t_0)$$