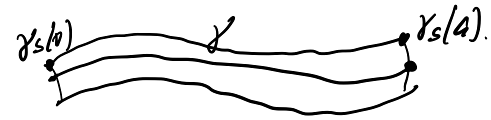
$$E(c) = \int_0^a |c'(t)|^2 dt, \quad c: [0, a] \to M$$

complete Riem. mfd.

For a smooth geodesic $\gamma: [0, a] \to M$, $\gamma: (-\varepsilon, \varepsilon) \times [0, a] \to M$

$(s, t) \mapsto \gamma(s, t)$.

$$V = \frac{\partial}{\partial s} \gamma(s, t) \Big|_{s=0}$$



$$\frac{d}{ds} E(s) \Big|_{s=0} = 0.$$

$$\boxed{\frac{1}{2} \frac{d^2}{ds^2} E(s) \Big|_{s=0} = \int_0^a \left(\langle V', V' \rangle - \underline{\langle R(\gamma', V) V, \gamma' \rangle} \right) dt + \underline{\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \gamma' \rangle (a)} - \underline{\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \gamma' \rangle (0)}.}$$

Thm (Bonnet–Myers) $Ric(g) \geq \frac{n-1}{r^2} g$, $\Rightarrow$ $diam(M) \leq \pi \cdot r$.

Thm (Weinstein, Synge) $f: M^n \to M$ is an isometry of a compact oriented Riem. mfd. $\boxed{sec(M) > 0}$

If n is even, assume f preserves the orientation.

If n is odd, assume f reverses the orientation.

Then f has a fixed point, i.e. $\exists\, p \in M$ s.t. $f(p) = p$.

Cor (Synge) M^n cpt. Riem. mfd. sec >0.

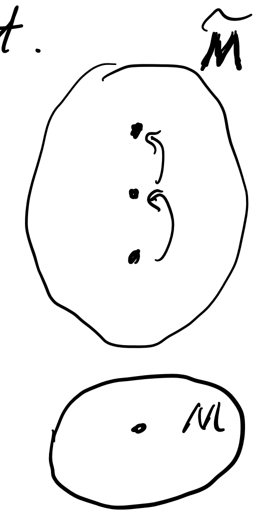
a) If n is even and M is orientable, then M is simply connected.

b) If n is odd, then M is orientable.

Pf a) universal covering $\pi: \widetilde{M} \to M$. $\tilde{g} = \pi^* g$ on $\widetilde{M}$

$\sec(\tilde{g}) > 0 \xRightarrow{\text{Bonnet-Myers}} \widetilde{M}$ is compact.

$\sec(\tilde{g}) = \pi^* \sec(g)$



$\pi_1(M) \circlearrowright \widetilde{M}$ as covering transformation (deck)

If M is not simply connected, then $\pi_1(M) \neq \{e\}$

$e \neq k \in \pi_1(M)$ $k: \widetilde{M} \to \widetilde{M}$ isometry

n is even $\xRightarrow{\text{Weinstein-Synge}}$ k has a fixed point. not possible for nontrivial covering trans. ✓

b) n is odd and suppose M is not orientable.

Then $\exists$ a double cover $\pi: \widetilde{M} \to M$ s.t. $\widetilde{M}$ is orientable

$\widetilde{M} = \{\text{frame bundle of } M\}/\sim$

$\{\text{frame bundle of } M\} = \{\text{basis of } T_pM, p \in M\}$

with a $\mathbb{Z}_2$-covering transformation

$\mathbb{Z}_2 = \{e, k\}$

$k: \tilde{M} \to \tilde{M}$ reverses the orientation of $\tilde{M}$.

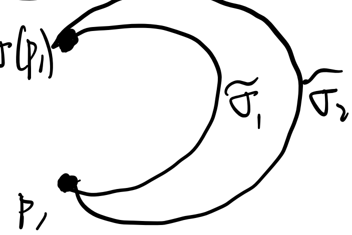
By Weinstein–Synge, k has a fixed point. not possible.

$\Longrightarrow$ M is orientable. $\blacksquare$

$$\boxed{\pi_1(M) = \{\overset{\text{pointed}}{\text{closed curves}}\}/\sim}$$

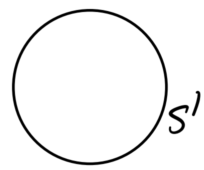
$$\begin{tikzcd} \tilde{M} \arrow[r, "k"] \arrow[dr, "\mu"'] & \tilde{M} \arrow[d, "\mu"] \\ & M = \tilde{M}/\pi_1(M) \end{tikzcd}$$

$k = [\sigma]$ $\quad k\cdot p = \sigma(p_1)$



$$\boxed{\mu\circ k = \mu}$$

Ex: $\mathbb{R} \xrightarrow{\exp} S^1$, $\quad \underline{\pi_1(S^1) = \mathbb{Z}}$ acting on $\mathbb{R}$



$(\mu\circ k)^* g = k^* \mu^* g \Rightarrow k$ is an isometry.

$\quad \| $

$\mu^* g$

<u>Ex</u>: $\mathbb{R}P^2 \times \mathbb{R}P^2 = \overset{\text{orientable}}{(S^2 \times S^2)}/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ does not admit Riem. metric g with sec>0.

$\sigma = \langle v, w \rangle$ $\quad$ sec≥ 0

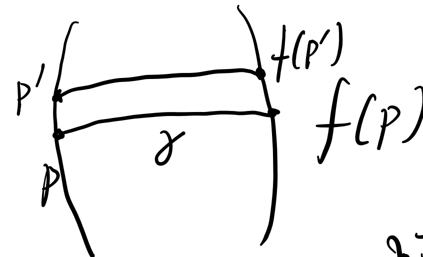
Big Open Question (Hopf) $\underline{\exists ?\ \text{Riem. } g \text{ sec}>0 \text{ on } S^2 \times S^2}$

pf of Weinstein-Synge : Proof by contradiction.

Suppose f does not have a fixed point. $f(p)\neq p \ \forall p\in M$

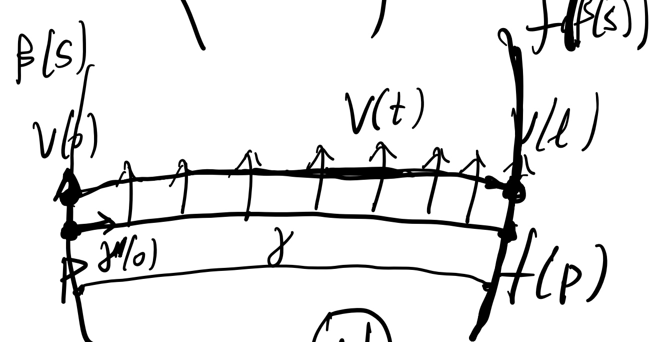
$q\mapsto \underline{d(q,f(q))}$. $\quad \min\limits_{q\in M} d(q,f(q)) = \underline{d(p,f(p))}$

continuous fct. on cpt. M.



γ : minimizing geodesic

$\gamma(0)=p,\quad \gamma(\underset{\substack{\shortparallel \\ L(\gamma)}}{l}) = f(p)$



Choose $V(0)\in T_pM$.

$\rightsquigarrow V(t) = P_{\gamma(0)}^{\gamma(t)}(V(0)) \in T_{\gamma(t)}M$

$$\begin{tikzcd} T_pM \arrow[r, "df"] \arrow[d, "\tilde{A}"'] & T_{f(p)}M \arrow[dl, "P_{f(p)}^{p}"] \\ T_pM & \end{tikzcd}$$

Want $V(0)$ satisfies: $|V(0)|=1$

$\tilde{A}(V(0)) = V(0)$. $\quad V(0)\perp\gamma'(0)$

$$\boxed{h(s,t) = \mathrm{Exp}_{\gamma(t)}(s\cdot V(t)).}$$

$h(s,0) = \mathrm{Exp}_{\gamma(0)}(s\,\underline{V(0)})$
$\shortparallel$
$\beta(s)$. $\uparrow$ geodesic

f is isometry

$\Rightarrow$ $\underline{f \circ \beta = \tilde{\beta}(s)}$ is a geodesic

$$\underline{\frac{d}{ds}(f \circ \beta)\Big|_{s=0}} = df(\beta'(0)) = df(V(0))$$

$$= P_p^{f(p)} \circ \big(\underbrace{P_{f(p)}^{p} \circ df}_{= \tilde{A}}\big)(V(0)) = P_p^{f(p)}(V(0)) = \underline{V(\ell)}$$

$\Rightarrow$ $f \circ \beta(s) = \mathrm{Exp}_{f(p)}(s \cdot V(\ell))$

$$\frac{d}{ds} E(s)\Big|_{s=0} = 0.$$

$E(s) = E(h(s, \cdot))$.

sec$(\langle \gamma', V \rangle)$.

$$\frac{1}{2}\frac{d^2}{ds^2}E(s) = \int_0^\ell \underbrace{\langle V', V' \rangle}_{=0} - \underline{\langle R(\gamma', V)V, \gamma' \rangle}\, dt < 0.$$

$$+ \underline{\langle \underbrace{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}}_{=0}, \gamma' \rangle(\ell)} - \underline{\langle \underbrace{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}}_{=0}, \gamma' \rangle(0)}.$$

$$\begin{tikzcd} T_pM \arrow[r, "df"] \arrow[d, "\tilde{A}"'] & T_{f(p)}M \arrow[dl, "P^{p}_{f(p)}"] \\ T_pM & \end{tikzcd}$$

$$\Rightarrow \tilde{A} : \underset{\mathbb{R}^n}{\overset{}{T_pM}} \cong \mathbb{R}^n, \quad \tilde{A}: T_pM \to T_pM$$

is an orthogonal linear transf.

$$\tilde{A}(\gamma'(0)) = \gamma'(0) \Rightarrow A = \tilde{A}\big|_{\gamma'(0)^\perp} : \gamma'(0)^\perp \to \gamma'(0)^\perp, \quad \gamma'(0)^\perp \cong \mathbb{R}^{n-1}$$

$$\det(A) = \det(\tilde{A}) = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

$$\Rightarrow \exists\ V(0) \in \gamma'(0)^\perp \text{ s.t. } \underline{A \cdot V(0) = V(0)}.$$

$$V(0) \neq 0, \quad \gamma'(0)^\perp \cong \mathbb{R}^{n-1}$$

If n is even, then $\underline{\det(A - \lambda I)}$ degree $(n-1)$-poly.

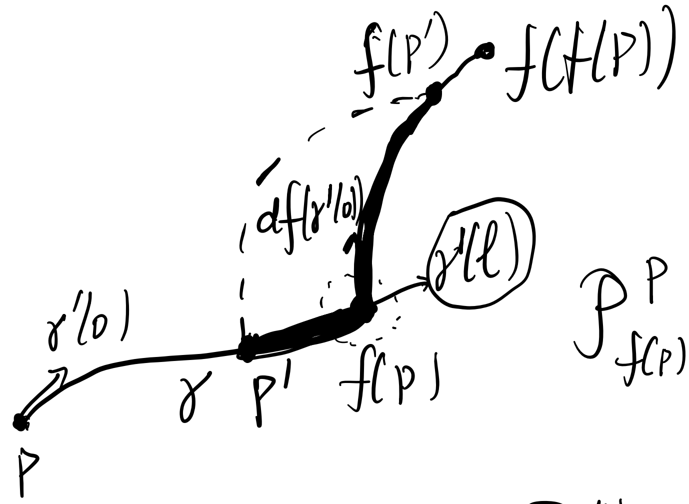
$(n-1)\times(n-1)$ matrix — odd

$\Downarrow$

1 is an eigenvalue

n is odd. $\det(A) = -1$

$(n-1)$ even.



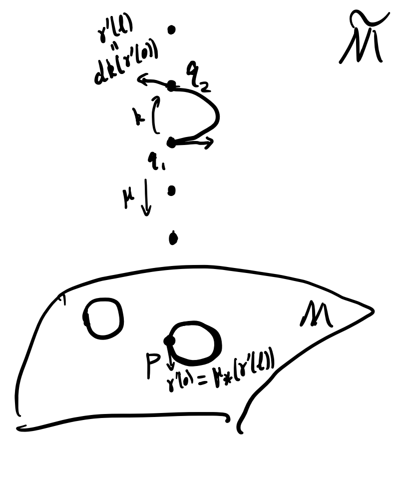
$$\tilde{A}(\gamma'(0)) = P^{p}_{f(p)} \circ \underbrace{df(\gamma'(0))}_{= \gamma'(l)} = \gamma'(0).$$

$$\nabla_{\gamma'}\gamma' = 0 \Rightarrow \gamma'(t) = P^{\gamma(t)}_{\gamma(0)}\,\gamma'(0).$$

$$\underbrace{d(p', f(p'))}_{= d(p, f(p))} \leq \underbrace{d(p', f(p)) + \underbrace{d(f(p), f(p'))}_{= d(p, p')}}_{d(p, f(p))}$$

$$\Rightarrow \boxed{df(\gamma'(0)) = \gamma'(l).} \quad \blacksquare$$

Synge:



geodesic loop