· $E: \Omega_{p,q} \longrightarrow \mathbb{R}$ $\qquad E(c) = \int_0^a |c'(t)|^2\, dt.$

$$\Omega_{p,q} = \left\{ \begin{array}{l} c:[0,a]\to M,\ \text{pcws. diff.}, \\ c(0)=p,\ c(a)=q \end{array} \right\}$$

critical points of E are smooth geodesics:

$$\frac{dE}{ds}\Big|_{s=0}(\underset{\displaystyle c(s,\cdot)}{\underset{\shortparallel}{c_s}}) = 0 \quad \text{for any proper variation of } c.$$

$c(s,t): (-\epsilon,\epsilon)\times[0,a]\to M$

$c(0,t) = c(t)$

$c(s,0)=p,\ c(s,a)=q$

$$\frac{1}{2}\frac{dE}{ds}(c_s)\Big|_{s=0} = \frac{1}{2}\frac{d}{ds}\int_0^a \langle \tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}\rangle dt = \frac{1}{2}\int_0^a 2\cdot\langle \underbrace{\nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial t}}_{=\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}}, \tfrac{\partial}{\partial t}\rangle dt$$

$$\langle \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}\rangle = \frac{\partial}{\partial t}\langle \tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}\rangle - \langle \underset{V}{\underset{\shortparallel}{\tfrac{\partial}{\partial s}}}, \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial t}\rangle$$

$$= -\int_0^a \langle V, \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial t}\rangle dt + \sum_i \langle V, \tfrac{\partial}{\partial t}\rangle\Big|_{t_i^+}^{t_{i+1}^-}.$$

$\rightsquigarrow$ $\frac{1}{2}\frac{dE}{ds}\Big|_{s=0} = 0$ $\forall$ proper variational vector field V ($V(0)=0,\ V(a)=0$) along c

$\Rightarrow$ $\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = 0$, c is C^1-differentiable $c'(t_i^-) = c'(t_i^+)$, $\forall i$

· 2nd variation for $\gamma: [0,a]\to M$ a geodesic.

$$\frac{1}{2}\frac{d^2}{ds^2}E(s) = \frac{d}{ds}\,\frac{1}{2}\frac{d}{ds}E = \frac{d}{ds}\int_0^a \langle \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}\rangle dt$$

$$= \int_0^a \Big(\underbrace{\langle \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}\rangle}_{= I} + \langle \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial t}\rangle \Big) dt$$

$$\nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \qquad \tfrac{\partial}{\partial s} = V$$

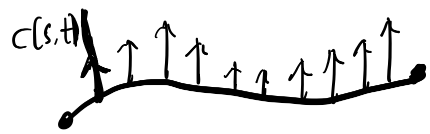
$$\langle \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\tfrac{\partial}{\partial s}\rangle = \langle \nabla_{\frac{\partial}{\partial t}}V, \nabla_{\frac{\partial}{\partial t}}V\rangle = \langle V', V'\rangle.$$

$$\underbrace{\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} - \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} - \underbrace{\nabla_{[\frac{\partial}{\partial s},\frac{\partial}{\partial t}]}\frac{\partial}{\partial s}}_{=0}}_{= R\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right)\frac{\partial}{\partial s}} + \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}$$

$$I = \int_0^a \left(\left\langle R\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right)\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle + \underbrace{\left\langle \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle}_{= \frac{\partial}{\partial t}\left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle - \underbrace{\left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right\rangle}_{\overset{s=0}{=}0}} \right) dt$$

$$\overset{s=0}{=} \int_0^a \langle R(V,\gamma')V, \gamma'\rangle dt + \left(\left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle\Big|_0^a \right)$$

$$V(s,t) \quad \underline{V(s,0)} \overset{V(s)}{=} \quad V(0,0)=0, \quad V'(0) \neq 0$$



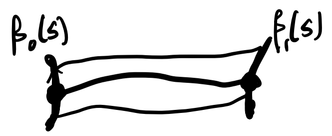
$$\boxed{\begin{aligned}\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} &= \int_0^a \left(\langle V',V'\rangle - \langle R(\gamma',V)V,\gamma'\rangle\right)dt \\ &\quad + \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \gamma'\right\rangle(a) - \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \gamma'\right\rangle(0).\end{aligned}}$$

Index form:

$$I_a(V,V) = \int_0^a \left(\langle V',V'\rangle - \langle R(\gamma',V)V,\gamma'\rangle\right)dt$$

$$I_a(V,W) = \int_0^a \left(\langle V',W'\rangle - \langle R(\gamma',V)W,\gamma'\rangle\right)dt$$

If $\begin{matrix}\nabla_{\frac{\partial}{\partial s}}V(0)=0 \\ \nabla_{\frac{\partial}{\partial s}}V(a)=0\end{matrix}$, then $\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} = I_a(V,V)$.



$V(s,0)=\beta_0'(s)$, $V(s,a)=\beta_1'(s)$.

$\nabla_{\frac{\partial}{\partial s}} V = \underline{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}}$

Boundary terms vanish if the end points varies along geodesics.

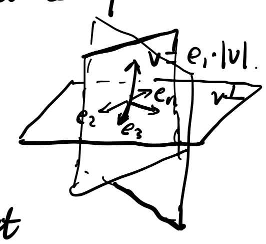
In particular they vanishes for proper variations (i.e. $c(s,0)=p$, $c(s,a)=q$)

· Thm (Bonnet-Myers) Let (M,g) be a complete Riem. mfd. Suppose $\text{Ric}(g) \geq \frac{n-1}{r^2} g$ for some $r>0$. Then M is compact and $\text{diam}(M,g) \leq \pi \cdot r$.

Rmk: $p \in M$, $v \in T_pM$, $v \neq 0$. Choose $\{e_i\}_{i=1}^n$ an. o.n.b. for T_pM.

$$\text{Ric}_p(v,v) = \sum_{i=1}^n \langle R(v,e_i)e_i, v\rangle$$ ← does not depend on the choice of o.n.b.

$$\text{Ric}_p(v) \qquad \boxed{\sum_{i=2}^n \langle R(v,e_i)e_i, v\rangle} \leftarrow$$



$$\text{Ric}_p(v,w) = \sum_{i=1} \langle R(v,e_i)e_i, w\rangle.$$

Bonnet

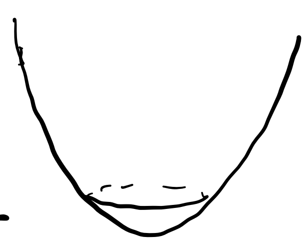
$$\sec \geq \varepsilon > 0 \Rightarrow \underline{\text{Ric}(v) \geq (n-1)\varepsilon|v|^2}, \ \forall v \xRightarrow{\text{Myers}} \text{diam}(M,g) \leq \frac{\pi}{\sqrt{\varepsilon}}$$

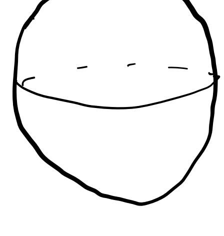
$$\text{Ric} \geq \frac{1}{r^2} g \Rightarrow \text{Ric}(v) \geq \frac{n-1}{r^2} |v|^2 \ \forall v \xRightarrow{\text{Bonnet-Myer}} \text{diam}(M,g) \leq \pi \cdot r$$

$(n-1)\varepsilon|v|^2 = \frac{n-1}{r^2}|v|^2$, $|v|^2 = g(v,v)$

parabola

$\sec > 0$





$$\mathrm{diam}(M,g) = \sup_{p,q\in M} d(p,q) \overset{M\text{ cpt.}}{=\!=} \underset{+\infty}{\underset{\wedge}{d(p,q)}} \quad \text{for some } p,q\in M$$

<u>Proof of Bonnet-Myer</u>: Enough to prove $\forall p,q\in M$, $\underline{d(p,q)\le \pi\cdot r}$.

Suppose not true. Then $\exists\, p,q$ s.t. $d(p,q) = L(\gamma) > \pi\cdot r$.

minimizing geodesic $\gamma(0)=p$, $\gamma(a)=q$.

γ is length minimizing $\Longrightarrow$ $\underline{E(\gamma)\le E(c)\ \ \forall c: p\rightsquigarrow q}$.

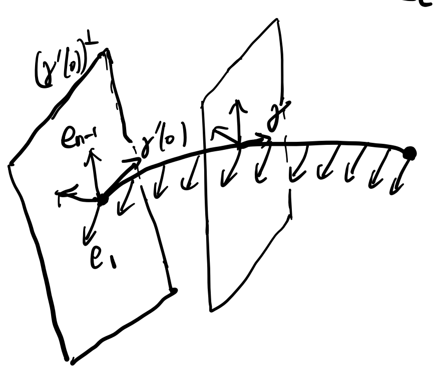
$$\boxed{\begin{array}{l}\gamma:[0,1]\to M \quad |\gamma'(t)|=L\\ \gamma(0)=p,\ \gamma(1)=q\end{array}}$$

Want to find proper variation $c(s,t)$ s.t. $\underbrace{\frac{1}{2}\frac{d^2}{ds^2}E(c_s)\Big|_{s=0}}_{E''(0)} < 0$.

$\frac{d}{ds}E(c_s)\big|_{s=0}=0 \rightsquigarrow$ $\overset{E(s)}{\overset{\shortparallel}{E(c_s)}} < E(\gamma)$ for $|s|\ll 1$

$\shortparallel$

$E(0) + \underline{E'(0)\cdot s} + \underline{\tfrac{1}{2}E''(0)\cdot s^2} + o(|s|^2)$



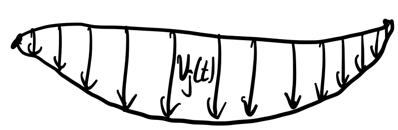
Choose o.n.b. $\{e_1, e_2, \dots, e_{n-1}\}$ for $\gamma'(0)^\perp \subset T_pM$

$\rightsquigarrow \{e_1(t), e_2(t), \dots, e_{n-1}(t)\}$ o.n.b. for $T_{\gamma(t)}M$.

$V_j(t) = \sin(\pi t)\, e_j(t)$, $\quad t\in[0,1]$

$P_t: T_pM \to T_{\gamma(t)}M$

<u>isomorphism of inner product spaces</u>



$$\frac{1}{2}\frac{d^2}{ds^2}E(s) = I(V,V) = \int_0^1 \left(\underline{\langle V',V'\rangle} - \langle R(\gamma',V)V,\gamma'\rangle\right)dt$$

$$\underset{\nabla_{\frac{\partial}{\partial t}}V_j}{\underset{\shortparallel}{V_j'(t)}} = \cos(\pi t)\cdot\pi\cdot e_j(t) + \sin(\pi t)\underline{\nabla_{\frac{\partial}{\partial t}} e_j(t)} = \pi\cos(\pi t)\,\underline{e_j(t)}$$

$$I_1(V_j,V_j)=\int_0^1 \left(\pi^2\cos^2(\pi t) - \underline{\langle R(\gamma', \sin(\pi t)e_j(t))\sin(\pi t)e_j(t), \gamma'\rangle}\right) dt$$

$$\parallel$$

$$\sin^2(\pi t)\,\underline{|\gamma'|^2}\left\langle R\left(\frac{\gamma'}{|\gamma'|}, e_j\right)e_j, \frac{\gamma'}{|\gamma'|}\right\rangle dt$$

$$\sum_{j=1}^{n-1} \frac{d^2}{ds^2}E_j(s)$$

$$\parallel$$

$$\sum_{j=1}^{n-1} I_1(V_j,V_j)=\int_0^1 (n-1)\pi^2\cos^2(\pi t) - \sin^2(\pi t)\cdot L^2\cdot \underline{Ric\left(\frac{\gamma'}{|\gamma'|}\right)}\ dt$$

$$\underline{L=d(p,q)} \qquad \geq \frac{n-1}{r^2}$$

$$\leq (n-1)\pi^2\int_0^1 \underline{\cos^2(\pi t)\,dt} - \frac{(n-1)L^2}{r^2}\int_0^1 \underline{\sin^2(\pi t)\,dt}.$$

$$\underline{\cos^2(\pi t)\,dt} = \frac{1+\cos(2\pi t)}{2}dt \qquad \underline{\sin^2(\pi t)\,dt} = \frac{1-\cos(2\pi t)}{2}$$

$$= (n-1)\pi^2\cdot\frac{1}{2} - \frac{(n-1)L^2}{r^2}\frac{1}{2} = \frac{(n-1)}{2}\cdot\left(\pi^2-\frac{L^2}{r^2}\right) \overset{L>\pi r}{<} 0. \blacksquare$$

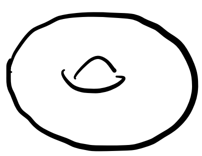
Cor: If $Ric(M,g)\geq (n-1)k>0$, then $|\pi_1(M)|<+\infty$.

Pf: $\begin{array}{c}\tilde{M}\\ \downarrow\pi\\ M\end{array}$ universal. $(\tilde{M}, \pi^*g)$ complete $\Rightarrow$ $\tilde{M}$ is compact.

$\Rightarrow$ π is a finite covering. $\forall p\in M$ $\#\{\pi^{-1}(p)\}<+\infty$

$$\#\{\pi^{-1}(p)\} = |\pi_1(M)|.$$

Ex:



$\pi_1(T^2)=\mathbb{Z}\oplus\mathbb{Z}$ $\quad |\pi_1(T^2)|=+\infty$

$\pi_1(T^n)=\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$.

can't support Riem. metric with positive Ricci curvature.

Q: Obstructions to the existence of
- positive Ricci ?
- positive sec ??
- positive scalar $\leftarrow$ good understanding

Thm (Synge) 1936: M compact, sec > 0

$\pi_1(M) = \{e\}$

$\Updownarrow$

(1) If M is even-dimensional and orientable, then M is simply connected

(2) If M is odd-dimensional, then M is orientable.

Idea of proof of (1): If not ($\pi_1(M) \neq \{e\}$), then find a closed $\overbrace{\text{geodesic}}^{\text{minimizing}}$ in a nontrivial homotopy class of $\pi_1(M)$