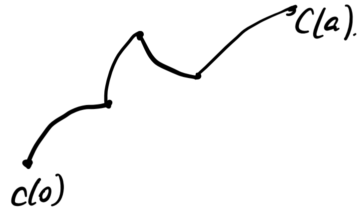
Variations of Energy and Applications

- $c : [0, a] \to M$ piecewise linear



$$E(c) = \int_0^a |c'(t)|^2 dt$$

$$L(c)^2 = \left(\int_0^a |c'(t)|\, dt\right)^2 \leq \underline{\left(\int_0^a 1^2 \cdot dt\right) \cdot \left(\int_0^a |c'(t)|^2 dt\right)}$$

$$= a \cdot E(c).$$

$$\left(\int f \cdot g \, dt\right)^2 \leq \int f^2 dt \int g^2 dt$$

"=" $\underset{\text{iff}}{\Leftrightarrow}$ $g = \lambda \cdot f$

"=" iff $\underline{|c'(t)| = \text{const.}}$ $\Leftrightarrow$ t is proportional to the length arc parameter

- Lemma: If $\gamma : [0, a] \to M$ is a <u>minimizing geodesic</u>, then $E(\gamma) \leq E(c)$ for any curve $c : [0, a] \to M$ s.t. $c(0) = \gamma(0)$, $c(a) = \gamma(a)$

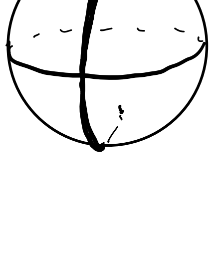
"=" holds $\Leftrightarrow$ c is also a minimizing geodesic.

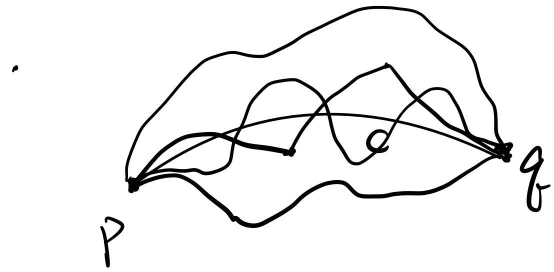
<u>Pf</u>: γ geodesic $\Rightarrow$ $|\gamma'| = \text{constant}$ $(\Leftarrow \nabla_{\gamma'} \gamma' = 0)$.

$$\Rightarrow a\, E(\gamma) = L(\gamma)^2 \underset{\gamma \text{ is minimizing}}{\leq} L(c)^2 \leq a \cdot E(c) \Rightarrow \underline{E(\gamma)} \leq E(c)$$

"=" $\Leftrightarrow$ $L(\gamma) = L(c)$ $\Rightarrow$ c is minimizing curve $\Rightarrow$ c is smooth geodesic

$L(c) = d(c(0), c(a))$





$$E : \underset{\parallel}{\Omega_{p,q}} \longrightarrow M$$

$\{ c : [0,a] \to M,\ c(0) = p,\ c(a) = q \}$
piecewise diff.

Proposition: $c : [0,a] \to M$ is a critical point of the energy functional w.r.t. proper variations if and only if c must be a geodesic.

$$\boxed{\frac{d}{ds} E(c_s)\Big|_{s=0} = 0}$$

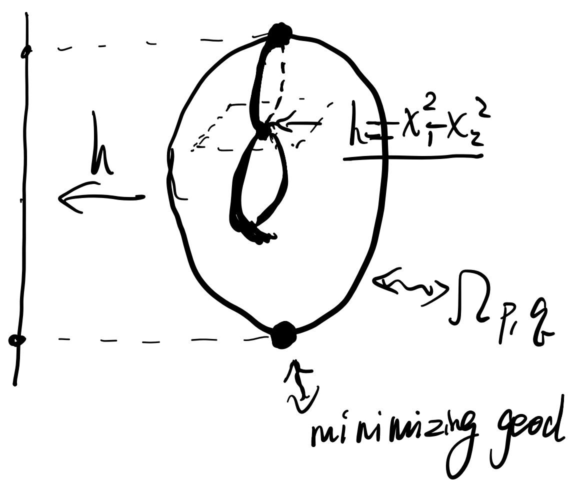
variation of c: $f : (-\varepsilon, \varepsilon) \times [0,a] \longrightarrow M$ piecewise differentiable

$$(s,t) \longmapsto f(s,t).$$

$$f(0,t) = c(t).$$

Proper variation: $f(s,0) = c(0)$, $f(s,a) = c(a)$

Prop: The Euler-Lagrange equation is the geodesic equation: $\nabla_{c'} c' = 0$.



$V(t)$: piecewise differentiable vector fields along $c:[0,a] \to M$

$V(t) \in T_{c(t)}M$

$$\underset{\parallel}{E(s)}$$

$$E(f(s,\cdot)) = \int_0^a \left|\frac{\partial f}{\partial t}\right|^2 dt$$

$$= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left|\frac{\partial f}{\partial t}\right|^2 dt$$

$$\frac{d}{ds}\frac{1}{2}E(s) = \frac{1}{2}\sum_{i=0}^{k} \frac{d}{ds}\int_{t_i}^{t_{i+1}} \left|\frac{\partial f}{\partial t}\right|^2 dt$$

$$\parallel$$

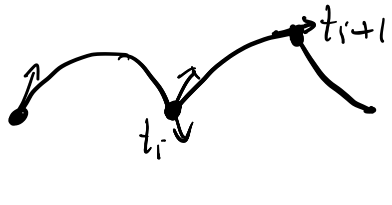
$$\int_{t_i}^{t_{i+1}} \underbrace{\frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle dt}_{\parallel} = \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle - \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right\rangle dt$$

$$\langle \underbrace{\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}}_{\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}}, \underbrace{\frac{\partial}{\partial t}}_{\frac{\partial f}{\partial t}} \rangle = \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right\rangle - \left\langle \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right\rangle$$

$$= \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right\rangle dt$$

$$\frac{d}{ds}\frac{1}{2}E(s)\Big|_{s=0} = \sum_i \langle V(t), c'(t)\rangle\Big|_{t_i^+}^{t_{i+1}^-} - \int_{t_i}^{t_{i+1}} \langle V(t), \nabla_{c'}c'\rangle dt.$$



(1st. variation formula)

If c is a smooth geodesic, then $\frac{d}{ds}E(s)\Big|_{s=0}=0$ for any proper variational vector field V.

$\Downarrow$

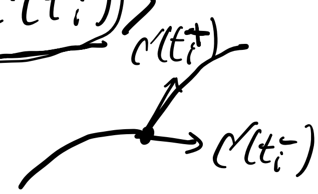
- $\nabla_{c'}c'=0$ on $[t_i, t_{i+1}]$, $\forall i$.

$$\Rightarrow \quad \frac{d}{ds}\frac{1}{2}E(s)\Big|_{s=0} = \sum_i \langle V(t), c'(t)\rangle\Big|_{t_i^+}^{t_{i+1}^-}$$

$$= \sum_{i=0}^{k}\left(\langle V(t_{i+1}), c'(t_{i+1}^-)\rangle - \langle V(t_i), c'(t_i^+)\rangle\right)$$

$$= \sum_{i=1}^{k}\langle V(t_i), c'(t_i^-)\rangle - \langle V(t_i), c'(t_i^+)\rangle$$

$$= \sum_{i=0}^{k}\langle \underbrace{V(t_i)}_{=-(c'(t_i^-)-c'(t_i^+))}, c'(t_i^-)-c'(t_i^+)\rangle$$



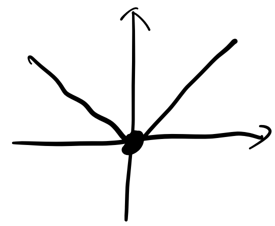
$$= -\sum_{i=0}^{k}\left|c'(t_i^-)-c'(t_i^+)\right|^2 \neq 0.$$

$\Rightarrow$ $c:[0,a]\to M$ is a C^1-geodesic.

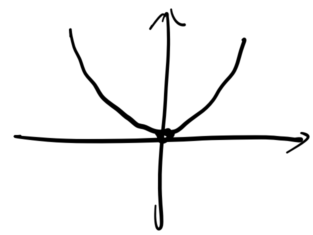
$\Rightarrow$ c is a smooth geodesic.

(Variational Characterization of Geodesics).

$L(c) = \int |c'(t)|\, dt$ $\qquad c(t(\tilde{t}))$ $\qquad |t|$



$$\frac{d}{ds} L(c_s) = \int_0^a \frac{\frac{\partial}{\partial s}|c'(t)|^2}{|c'(t)|} \cdot \frac{1}{2}\, dt$$



• $\gamma : [0, a] \to M$ a smooth geodesic. $\iff \frac{d}{ds} E(s)\Big|_{s=0} = 0$.

$$\frac{1}{2}\frac{d}{ds} E(s) = \frac{1}{2}\frac{d}{ds} \sum_i \int_{t_i}^{t_{i+1}} \Big|\overset{\frac{\partial t}{\partial t}}{\overset{\|}{\frac{\partial}{\partial t}}}\Big|^2 dt$$

$$= \sum_i \int_{t_i}^{t_{i+1}} \Big\langle \underbrace{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}}_{= \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}}, \frac{\partial}{\partial t} \Big\rangle dt$$

$V = \frac{\partial}{\partial s}$

$\frac{\partial}{\partial s} \Big\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \Big\rangle$ $\qquad \nabla_{\frac{\partial}{\partial t}} V = V'$

$$\frac{1}{2}\frac{d^2}{ds^2} E(s) = \sum_i \int_{t_i}^{t_{i+1}} \Big(\Big\langle \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \Big\rangle + \Big\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}, \underset{\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} = V'}{\underset{\|}{\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}}} \Big\rangle \Big) dt$$

$$\|$$

$$\boxed{\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} - \nabla_{\underset{0}{\underset{\|}{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]}}} \frac{\partial}{\partial s}}$$

$$+ \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}$$

$$= \sum_i \int_{t_i}^{t_{i+1}} \Big(\Big\langle R\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big) \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \Big\rangle + \underbrace{\Big\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \Big\rangle}_{\| } + \boxed{\langle V', V' \rangle} \Big) dt$$

$$\frac{\partial}{\partial t} \Big\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \Big\rangle - \Big\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \Big\rangle$$

$$\frac{1}{2}\frac{d^2}{ds^2}E(s)\Big|_{s=0} = \sum_{i}\int_{t_i}^{t_{i+1}} \underline{\langle R(V,\gamma')V, \gamma'\rangle + \langle V', V'\rangle}$$

$$+ \underline{\sum_{i=0}^{k} \langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\rangle\Big|_{t_i}^{t_{i+1}}}$$

$$\|$$

$$\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\rangle\Big|_{t=a} - \langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\rangle\Big|_{t=0}$$