Fact 1: / **Lemma**

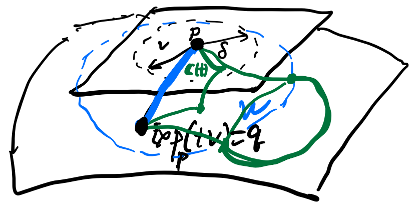
If $\mathrm{Exp}_p : \underset{T_pM}{U \cap} \to M$ is a diffeomorphism on $B_\delta(p) \subset U \subset T_pM$

onto $W = \mathrm{Exp}_p(B_\delta(p)) \subset M$, then

for any $v \in B_\delta(p)$, $\gamma_v(t) = \mathrm{Exp}_p(tv)$

is the unique geodesic that has the

minimal length among all curves connecting p to $\mathrm{Exp}_p(v) = q$.



Proof: Special case: when $\{c(t)\} \subset W = \mathrm{Exp}_p(B_\delta(p))$

$$c(t) = \mathrm{Exp}_p(\underbrace{w(t)}_{\in B_\delta(p)}) = \mathrm{Exp}_p(r(t)\cdot v(t)) \qquad |v(t)| = 1, \quad v(t) = \frac{w(t)}{|w(t)|}, \quad r(t) = |w(t)|$$

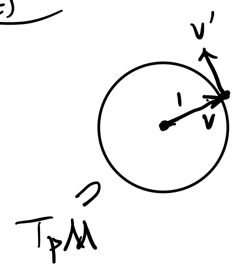
$t \mapsto v(t) \in S(p) = \{v \in T_pM, |v| = 1\}$ piecewise differentiable

$$c'(t) = \frac{d}{dt}\underbrace{\mathrm{Exp}_p(r(t)\cdot v(t))}_{f(r,v),\ r = r(t),\ v = v(t)} = \frac{\partial f}{\partial r} r'(t) + (d\mathrm{Exp}_p)_{r(t)\cdot v(t)}(r(t)\cdot v'(t))$$

$$= (d\mathrm{Exp}_p)_{r(t)\cdot v(t)}\left(\frac{d}{dt}(r(t)\cdot v(t))\right)$$

$$= \underbrace{(d\mathrm{Exp}_p)_{r\cdot v}(r'\cdot v)}_{u_1 \,=\, \gamma'(t)} + \underbrace{(d\mathrm{Exp}_p)_{r\cdot v}(r\cdot v')}_{u_2}$$

$$T_pM \xrightarrow{d\mathrm{Exp}_p} T_{c(t)}M \qquad T_{c(t)}M$$



$v' \perp v \Leftarrow |v|^2 = 1$, $\langle v', v\rangle = 0$

Fact (Gauss) $u_1 \perp u_2$, $|u_1|^2 = |r'v|^2$

$$\Rightarrow |c'(t)|^2 = |u_1|^2 + |u_2|^2 = |r'v|^2 + \underbrace{|u_2|^2}_{\geq 0} \geq |r'|^2$$

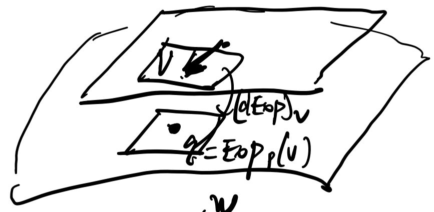
$$\Rightarrow |c'(t)| \geq |r'| \Rightarrow \int_0^a |c'(t)|\,dt \geq \int_0^a |r'(t)|\,dt \geq \int_0^a r'(t)\,dt = r(a) - r(0) = L(\gamma(t)).$$

$\mathrm{Exp}_p(r(a)\cdot v(a)) = q$, $r(a) = L(\gamma(t))$

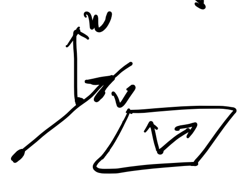
Lemma (Gauss) $\forall v$

$$\langle (d\mathrm{Exp}_p)_v(\overset{a}{\underset{w'}{v}}), (d\mathrm{Exp}_p)_v(\overset{b}{w}) \rangle = \langle \overset{a}{v}, \overset{b}{w} \rangle$$

$$\mathrm{Exp}_p : T_pM \longrightarrow M$$
$$v \longmapsto \mathrm{Exp}_p(v) = q$$



$$(d\mathrm{Exp}_p)_v : T_v(T_pM) \longrightarrow T_qM.$$
$$T_v(T_pM) \cong T_pM$$



$$\boxed{J'(0) = w}$$

$$(d\mathrm{Exp}_p)_{tv}(tw) = J(t) = \frac{\partial}{\partial s}(\mathrm{Exp}_p)(t(v+sw))\Big|_{s=0}$$

$$(d\mathrm{Exp}_p)_{tv}(v) = \gamma'(t) = \frac{d}{dt}(\mathrm{Exp}_p(tv))$$

$$\boxed{\langle \gamma'(1), J(1) \rangle \overset{?}{=} \langle \gamma'(0), J'(0) \rangle}$$

$$f(t) = \langle \gamma'(t), J(t) \rangle, \quad f(0) = 0.$$

$$f'(t) = \frac{d}{dt}\langle \gamma', J \rangle = \langle \underset{0}{\gamma''}, J \rangle + \langle \gamma', J' \rangle = \langle \gamma', J' \rangle$$

$$f''(t) = \frac{d}{dt}\langle \gamma', J' \rangle = \langle \underset{0}{\gamma''}, J' \rangle + \langle \gamma', J'' \rangle$$

$$\langle \gamma', J'' \rangle = \langle \gamma', R(\gamma', J)\gamma' \rangle = 0.$$

$$\Rightarrow f(t) = b \cdot t. \Rightarrow f(1) = b$$

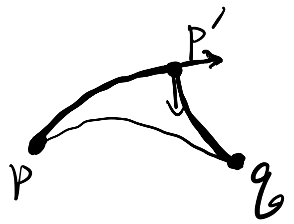
$$b = f'(0) = \langle \gamma'(0), J'(0) \rangle = \langle v, w \rangle = f(1)$$

Fact (Lemma)

$p \overset{\gamma}{\frown} q$ piecewise ~~differentiable~~ $\quad L(\gamma) = d(p, q) \;\Big(= \inf_{\substack{c(0)=p \\ c(a)=q}} L(c) \quad c \text{ piecewise diff.}\Big)$

$\Rightarrow$

γ must be a smooth (minimizing) geodesic.



~~total~~ normal geodesic ball

$W \subset \mathrm{Exp}_p(B_\delta(p))$
diff. image.

$\boxed{\text{Hopf-Rinow}}$

geodesic complete $\Longleftrightarrow$ geodesic complete at some $p \in M$

$\mathrm{Exp}_p : T_pM \to M$ is well defined

$\mathrm{Exp}_p(tv) : [0, +\infty) \to M, \ \forall v \in T_pM.$

$\Updownarrow$

bounded and closed subset is compact
(Heine-Borel property)

$\Updownarrow$

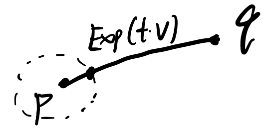
metrically complete (i.e. any Cauchy sequence converges).

Hadamard Thm: Let (M^n, g) be complete, simply connected with $\sec \leq 0$. Then $\underline{\mathrm{Exp}_p : T_pM \to M \text{ is a diffeom.}}$

M is diffeomorphic to $\mathbb{R}^n$.

<u>Pf</u>: (1) Hopf-Rinow: $\mathrm{Exp}_p : T_pM \to M$ well-defined surjective

$\Longrightarrow$ M is metrically complete.



(2) $\sec \leq 0 \Longrightarrow (d\mathrm{Exp}_p)_v : \underset{T_v(T_pM)}{\underset{\parallel}{T_pM^n}} \to T_{\underset{q}{\underset{\parallel}{\mathrm{Exp}_p(v)}}} M^n$ is invertible.

$w \neq 0 \in T_pM$, $\underline{(d\mathrm{Exp}_p)_v(w) = J(1)}$. $J(t) = (\mathrm{Exp}_p)_{*tv}(tw)$.

$\sec \leq 0 \Longrightarrow |J(t)| \neq 0$. $f(t) = |J(t)|^2$

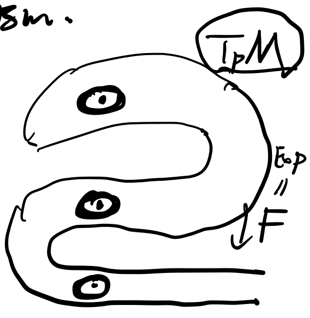
$(f'' \geq 0,\ f'(0) > 0)$

$\xrightarrow{\text{Inverse Function Thm.}}$ Exp_p is a local diffeomorphism.

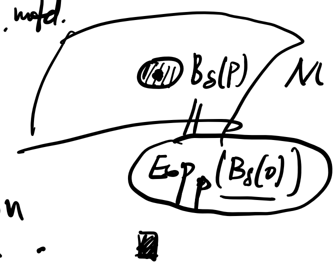
(3) $\overset{F}{\overset{\parallel}{\mathrm{Exp}_p}} : (T_pM, G) \to (M, g)$ is a $\underline{\text{covering map}}$.

Construct a Riem. metric on $\underline{T_pM}$: $\boxed{F^*g = G}$

$\uparrow$ complete Riem. mfd.

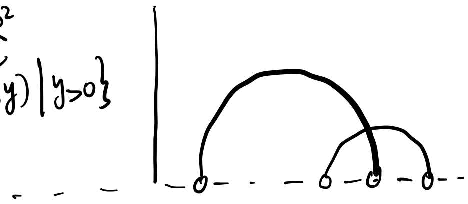


$\underset{F}{\underset{\parallel}{(\mathrm{Exp}_p)^{-1}}}\left(\underline{B_\delta(p)}\right) = \bigsqcup_{q \in F^{-1}(p)} B_\delta(q)$ w.r.t. G



(4) M is simply connected $\Longrightarrow$ $T_pM \cong M \cong \mathbb{R}^n$. ∎

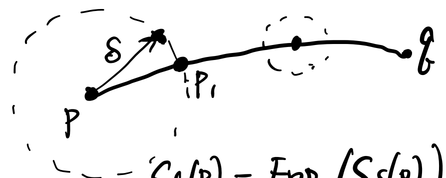
$\mathbb{H}^2 = \{(x,y) \mid y>0\}$ ($\Downarrow \mathbb{R}^2$)



$\overset{\text{diff.}}{\cong} \mathbb{R}^2$

$\underline{Exp_p : T_pM \to M \text{ is defined}} \Rightarrow \forall q \in M,\ \exists v \in T_pM$ s.t. $Exp_p(tv) = \gamma(t)$ is a minimizing geodesic connecting p to q.

$A = \{t \in [0, L] \mid d(\gamma(t), q) = L - t\}$



$\underline{S_\delta(p) = Exp_p(S_\delta(0))}$

$d(p_1, q) = d(q, S_\delta(p))$

$p_1 = Exp_p(\delta \cdot v_1)$

$\gamma(t) = Exp_p(tv_1)$. $\quad d(\gamma(t), q) = d(p,q) - t$.

$\forall t \in [0, +\infty)$ $\quad d(\gamma(L), q) = d(p,q) - L = 0 \Rightarrow \gamma(L) = q$.

- $\underline{sec \geq \varepsilon > 0}$ $\Downarrow$ $Ric \geq \varepsilon > 0$ $\xrightarrow[\text{Bonnet-Meyer}]{M \text{ complete}}$ M is compact. $diam(M) < +\infty$.

- $\underline{sec > 0}$ $\xrightarrow[M \text{ open (not closed)}]{\text{Gromoll-Meyer}}$ $M \underset{\text{diff.}}{\cong} \mathbb{R}^n$

- $\sec \geq 0$. $\xRightarrow[\text{(Soul Thm)}]{\text{Cheeger–Gromoll}}$ $M \overset{\text{diff.}}{\cong}$ vector bundle over a $\underbrace{\text{totally geodesic submfd}}_{\substack{\| \\ \text{Soul of } M}}$

$\left.\begin{array}{l}\sec \geq 0 \text{ on } M \\ \sec > 0 \text{ at some } p \in M\end{array}\right\} \xRightarrow{\text{Perelman}} M \underset{\text{diff.}}{\cong} \mathbb{R}^n$.