Thm (Hopf–Rinow) Let (M,g) be a $\underbrace{\text{Riem. mfd}}_{\text{connected}}$. The following conditions are equivalent:

(1) (M,g) is geodesically complete: $\underline{\forall p \in M,\ \mathrm{Exp}_p \text{ is defined on } T_pM}$

(2) $\underline{\exists p \in M}$ s.t. Exp_p is defined on T_pM.

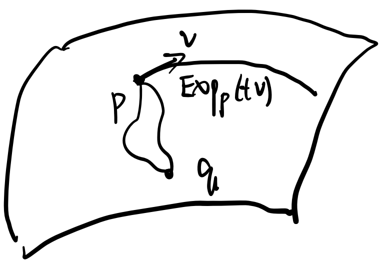
(3) (M, d_g) satisfies the Heine-Borel property: any closed and bounded subset of M is compact

(4) (M, d_g) is metrically complete.

For this given p

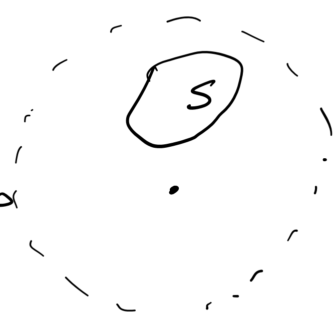
Moreover, if (M,g) satisfies any of these conditions, then

(5) $\underline{\forall p, q \in M, \text{ there exists a geodesic } \gamma \text{ from } p \text{ to } q \text{ s.t. } L(\gamma) = d_g(p,q).}$



Geodesic complete at $p \in M$:
$\mathrm{Exp}_p(tv)$ is defined for any p, any v and any t

$$p, q \in M, \quad d_g(p,q) = \inf_{\substack{\gamma:[0,a]\to M \\ \gamma(0)=p,\ \gamma(a)=q \\ \gamma \text{ piecewise differentiable}}} L(\gamma).$$ Metric structure

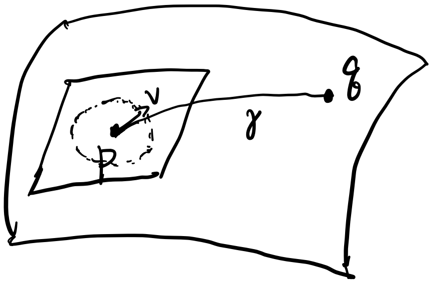


- $S \subset M$ is bounded, $\sup_{q \in S} d_g(p,q) < +\infty$
- metrically complete: any Cauchy sequence has a (unique) limit.

$(2)\Rightarrow(3)$: Assume $\text{Exp}_p : T_pM \to M$ is defined for any $v\in T_pM$ for some $p\in M$

Want Show: $\forall q\in M$, $\exists$ a geodesic $\gamma:[0,a]\to M$, $\gamma(0)=p$, $\gamma(a)=q$

s.t. $L(\gamma)=d(p,q)$ $\quad (d=d_g)$.

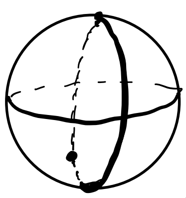


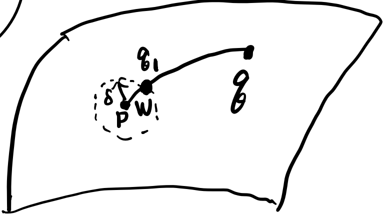
$\gamma(t)=\text{Exp}_p(tv)$

Problem is to find such a $v\in T_pM$.

Fact (Lemma) $\exists$ small ball $\overline{B_\delta(p)}\subset T_pM$ s.t. $\text{Exp}_p|_{B_\delta(p)} : \overline{B_\delta(p)} \to W\subset M$ is a diffeomorphism and $\underbrace{d(p,\text{Exp}_p(v))}_{=L(\{\text{Exp}_p(tv),\ 0\le t\le 1\})}=|v|$, $\forall v\in \overline{B_\delta(p)}$.

$$\text{Exp}_p|_{S_\delta(p)} : S_\delta(p)\to \partial W\subset M, \quad S_\delta(p)=\partial B_\delta(p)$$



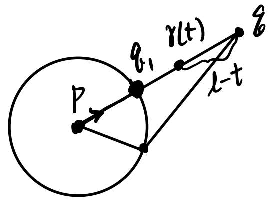


$\forall q'\in \partial W$, $d(p,q')=\delta$

$\partial W \Updownarrow \text{Exp}(S_\delta(p))$

$\inf_{q'\in\partial W} d(q,q')$

- $\exists q_1\in\partial W$, s.t. $d(q_1,q)=d(q,\partial W)$, $\quad \partial W\cong S_\delta(p)\cong S^{n-1}$



- $q_1=\text{Exp}(\delta v_1)$ for a unique $v_1\in T_pM$, $|v_1|=1$
- Need to show $\gamma(t)=\text{Exp}_p(tv)$, $t\in[0,+\infty)$ satisfies $\gamma(0)=p$, $\gamma(l)=q$ where $l=d(p,q)$.

$\rightsquigarrow$ Need to show $\boxed{d(\gamma(t), q) = \ell - t}$ $\forall\, t \in [0, \ell]$

- Continuity method: $\underline{A = \{t \in [0,\ell] : d(\gamma(t), q) = \ell - t \text{ holds true}\} \subseteq [0,\ell]}$

A is nonempty : $t \in [0, \delta] \subseteq A$

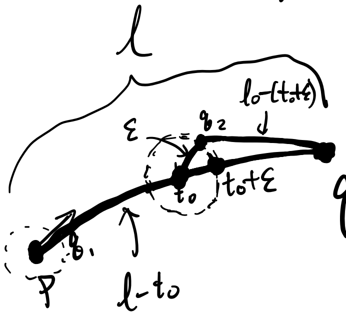
$$d(\gamma(t), q) \geq d(p,q) - d(p, \gamma(t)) = \ell - t.$$

$$d(\gamma(t), q) \leq d(\gamma(t), q_1) + d(q_1, q) = \underbrace{(\delta - t) + \ell - \delta}_{= \ell - t}.$$

$\sup A = \ell. \Rightarrow \ell \in A \Rightarrow \gamma(\ell) = q$

$\ell = d(p,q)$.

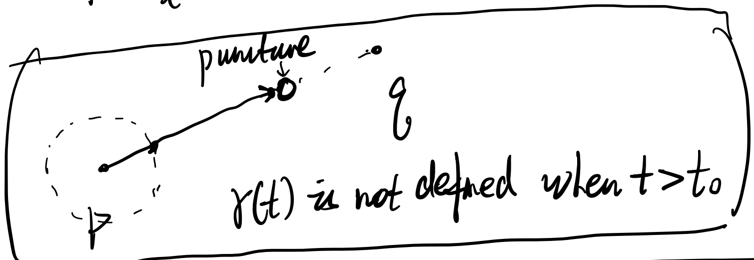
$\sup A$ $\Uparrow$ closed A



$$\boxed{d(\gamma(t_0), q) = \ell - t_0 \overset{?}{\Rightarrow} \exists\, \varepsilon \text{ s.t. } t_0 + \varepsilon \in A}$$

$$\Downarrow$$

$$\sup A = \ell.$$



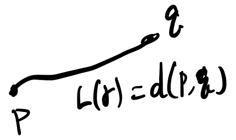
$\boxed{q_2 \in S_\varepsilon(\gamma(t_0)) \text{ s.t. } d(q_2, q) = \ell - t_0 - \varepsilon}$ $\quad q_2 \overset{?}{=} \gamma(t_0+\varepsilon)$

$$\underline{d(\gamma(t_0+\varepsilon), q)} \geq d(\gamma(t_0), q) - \underline{d(\gamma(t_0), \gamma(t_0+\varepsilon))} = \ell - t_0 - \varepsilon$$

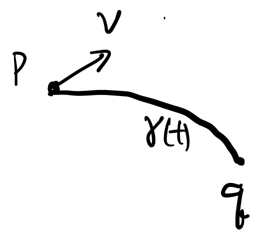
$$\underbrace{d(p,q)}_{= \ell} = \underbrace{L(p, \gamma(t_0))}_{= t_0} + \underbrace{L(\gamma(t_0), q_2)}_{= \varepsilon} + \underbrace{L(q_2, q)}_{= \ell - (t_0+\varepsilon)}$$

$\Longrightarrow$ Broken geodesic is regular (smooth). because it achieves the distance between the end points. $\Rightarrow q_2 = \gamma(t_0 + \varepsilon)$.

Fact (Lemma): $p, q \in M$, if a piecewise differentiable curve γ from p to q satisfies $\underline{L(\gamma) = d(p,q)}$. Then γ is a smooth (regular) geodesic.



$\Rightarrow \gamma$ sm. geodesic.



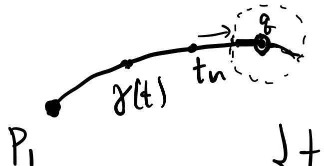
$Exp(tv)$ defined $\forall t \in [0, +\infty)$ $\forall v \in T_pM$

(3) closed and bounded subset S is compact.

$\exists \; r >> 1$, s.t. $S \subseteq \underline{Exp_p(\overline{B_r(p)})}$, $\overline{B_r(p)} \subset T_pM$

$Exp_p(v)$ $|v| \le r$

(4) $\Downarrow$ (1)



$\underline{d(\gamma(t_m), \gamma(t_n))} \le |t_m - t_n| < \varepsilon$

$\{t_m\} \subset [0, \underline{A})$ Cauchy sequence

$\Rightarrow \{\gamma(t_n)\}$ Cauchy sequence $\xRightarrow{(4)}$ $\gamma(t_n) \longrightarrow q \in M$ ∎