Let $\gamma : [0,\ell] \to M$ be a geodesic.

Jacobi field: J a vector field along γ ($J(t) \in T_{\gamma(t)}M$) satisfies

$$\boxed{J'' = R(\gamma', J)\gamma'}$$

$\{J(t) : 0 \le t \le \ell\}$ is uniquely determined by $J(0)$, $J'(0) \in T_pM$

Prop: Assume $J(0)=0$. $|w|=1$. $w = J'(0)$.

$$f(t) = \underset{\underset{(J,J)}{\parallel}}{|J(t)|^2} = t^2 - \frac{1}{3}\big(R(\underset{\underset{\gamma'(0)}{\parallel}}{v}, \underset{\underset{J'(0)}{\parallel}}{w})w, v\big)\, t^4 + o(t^4)$$

Taylor expansion near $t=0$.

Proof: $\frac{d}{dt}(J,J) = (\nabla_{\frac{\partial}{\partial t}} J, J) + (J, \nabla_{\frac{\partial}{\partial t}} J) = 2(J', J) \overset{t=0}{\Rightarrow} f'(0) = 0$

$$\frac{d^2}{dt^2}(J,J) = \frac{d}{dt}\, 2(J',J) = 2\big((J'',J) + (J',J')\big) \Rightarrow f^{(2)} = |J'(0)|^2 2 = 2|w|^2$$

$$\frac{d^3}{dt^3}(J,J) = 2\big((J''',J) + (J'',J') + 2(J'',J')\big) = \underbrace{2(J''',J) + 6(J'',J')}_{\overset{t=0}{\parallel}\ 0}$$

$$f^{(4)}(0) = \frac{d^4}{dt^4}(J,J)\Big|_{t=0} = 2(J'''(0), J'(0)) + 6(J'''(0), J'(0)) + \underbrace{6(J''(0), J''(0))}_{\parallel\ 0}.$$

$$= 8(J'''(0), J'(0))$$

$$\Big(J''(0) = R(\gamma', \underset{\underset{0}{\parallel}}{J(0)})\gamma' = 0\Big)$$

$$J''' = \frac{d}{dt} J'' = \frac{d}{dt}\big(R(\gamma',J)\gamma'\big)$$

$$(J''', W) = \big(\nabla_{\frac{\partial}{\partial t}}(R(\gamma',J)\gamma'), W\big) = \frac{d}{dt}\underset{\underset{\frac{d}{dt}(R(\gamma',W)\gamma', J)}{\parallel}}{(R(\gamma',J)\gamma', W)} - (R(\gamma',J)\gamma', W')$$

$$= \langle \nabla_{\frac{\partial}{\partial t}} R(\gamma',W)\gamma', J\rangle + \langle R(\gamma',W)\gamma', J'\rangle - \langle R(\gamma',J)\gamma', W'\rangle$$

$$(J'''(0), W) \underset{\forall W}{=} \langle R(\gamma',W)\gamma', J'(0)\rangle = \langle R(\gamma', J'(0))\gamma', W\rangle \quad \big(\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle\big)$$

$$\Rightarrow \quad J'''(0) = R(\gamma', J'(0))\gamma' \qquad\qquad -8\langle R(v,w)w, v\rangle$$

$$f^{(4)}(0) = 8\langle R(\gamma', J'(0))\gamma', J'(0)\rangle = \underline{8\langle R(v,w)v, w\rangle}$$

$$f'(0)=0, \quad f''(0) = 2|w|^2 = 2\cdot 1 = 2. \quad \underline{f'''(0)=0}$$

$$f(t) = 0 + 0\cdot t + \frac{2}{2}\cdot t^2 + 0\cdot t^3 + \frac{-8\langle R(v,w)w,v\rangle}{4!}t^4 + o(t^4)$$

$$= t^2 - \frac{1}{3}\langle R(\underset{v}{\gamma'(0)}, \underset{w}{J'(0)})\underset{w}{J'(0)}, \underset{v}{\gamma'(0)}\rangle t^4 + o(t^4).$$

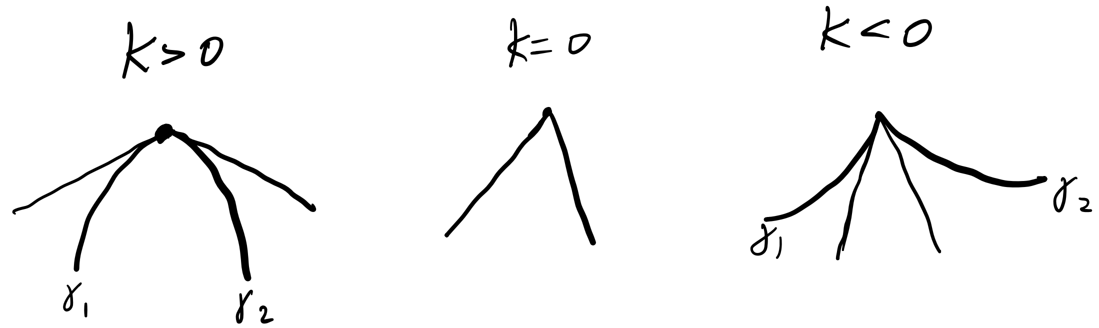
■

$$\underset{\substack{\| \\ \mathrm{Span}\{v,w\}\subset T_pM}}{\overset{K(p,\sigma_{v,w})}{\overset{\|}{\mathrm{sec}}}(\sigma_{v,w})} = \frac{\langle R(v,w)w,v\rangle}{|v|^2|w|^2 - \langle v,w\rangle^2} \overset{\{v,w\}\ \text{o.n.}}{=\!=} \langle R(v,w)w,v\rangle$$

$$\underset{|v|}{|\gamma'(0)|} = 1, \quad |\underset{J'(0)}{w}| = 1 \Rightarrow |J(t)|^2 = t^2 - \frac{1}{3}K(p,\sigma)t^4 + o(t^4).$$

$$\Downarrow$$

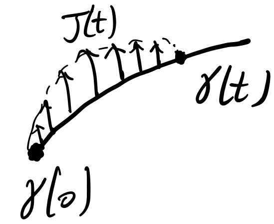
$$|J(t)| = t - \underline{\frac{1}{6}K(p,\sigma)t^3} + o(t^3)$$



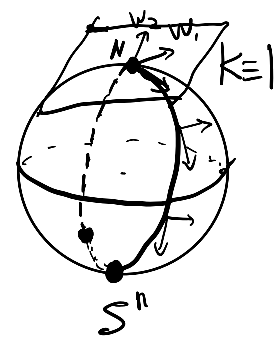
• **Def:** Fix $p \in M$. $\gamma : [0, a] \to M$ geodesic $\gamma(0) = p$, $t \in [0,a]$

point $\gamma(t)$ is conjugate to $\gamma(0)$ if there ~~exists~~ a Jacobi field $J \not\equiv 0$ $J(0) = 0 = J(t_0)$.

$$\text{Multiplicity of } \gamma(t_0) = \underbrace{\dim \{ J \not\equiv 0, J(0) = 0 = J(t_0) \}}_{\leq n-1}$$



Ex:



$K \equiv 1$

$J(t) = \sin t \cdot w(t) = 0$ when $t = k \cdot \pi, k \in \mathbb{N}$

$$\underbrace{(t \cdot \gamma'(t))''}_{J_1} = (\gamma' + t \cdot \underbrace{\gamma''}_{0})' = \gamma'' = 0 = R(\gamma', t\gamma')\gamma'$$

$J_1(0) = 0$, $|J_1(t)|^2 = |t|^2 |\gamma'(t)|^2 = |t|^2 |\gamma'(0)|^2 \neq 0$ if $t \neq 0$.

$$\dim \{ J \mid J(0) = 0 \} = n. \quad (t\gamma'(t) \in)$$

$$\dim \ker (\{ J \mid J(0) = 0 \} \to T_{\gamma(t_0)} M), \quad J \mapsto J(t_0)$$

$$\operatorname{mult}(\gamma(t_0)) = \dim \{ J \mid J(0) = 0 = J(t_0) \}. \quad t_0 \cdot (d\exp_p)_{t_0 \gamma'(0)}(w)$$

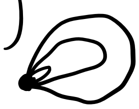
$$(t_0 \neq 0) \quad = \dim \{ (d\exp_p)_{t\gamma'(0)}(t \cdot w) \mid (d\exp_p)_{t_0 \gamma'(0)}(t_0 \cdot w) = 0 \}$$

$J(t) \Leftrightarrow w = J'(0)$

$= \dim \operatorname{Ker} \{ (d\exp_p)_{t_0\gamma'(0)} : T_pM \longrightarrow T_{\gamma(t_0)}M \}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\!\mid \; w$

- <u>Thm (Hadamard)</u> Let M^n be a Riemannian mfd. with $K \leq 0$ (K = sec)

<u>Assume</u>: • M is simply connected. ($\pi_1(M) = \{e\}$) 

• <u>M is complete</u> (as a metric space)

<u>Then</u>: M is diffeomorphic to $\mathbb{R}^n$.

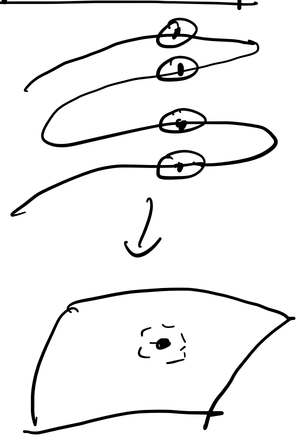
<u>Idea of proof</u>: $\boxed{\operatorname{Exp}_p : U \longrightarrow M}$, $U \subset T_pM$, $\operatorname{Exp}(U) \subset M$

A: • M is complete $\overset{\text{Hopf-Rinow}}{\Longleftrightarrow}$ Exp_p is defined on the whole T_pM

B: $\boxed{\bullet \; K \leq 0 \Longrightarrow \operatorname{Exp}_p \text{ is a local diffeomorphism}}$

C: • Exp_p is a <u>covering map</u>

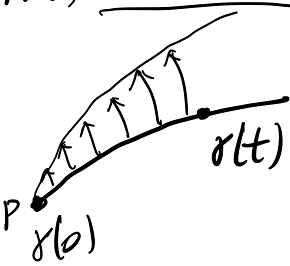
$\xRightarrow{M \text{ is simply connected}}$ Exp_p is a diffeomorphism.



Lemma: $K = \sec \leq 0$ $\quad \forall p \in M, \ \forall \sigma \subset T_pM$

$\sigma = \sigma_{v,w} = \mathrm{Span}\{v, w\}$

Then along any geodesic $\gamma : [0, a] \to M$, there is NO conjugate points to $\gamma(0)$.



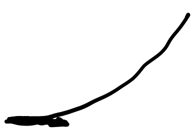
Pf: $f(t) = |J(t)|^2$, $\quad J(0) = 0$

$$f' = \frac{d}{dt}\langle J, J\rangle = 2\langle J', J\rangle, \quad f'(0) = 0$$

$$f'' = 2\langle J'', J\rangle + 2\langle J', J'\rangle = 2\underbrace{\langle R(\gamma', J)\gamma', J\rangle}_{= -2\langle R(\gamma', J)J, \gamma'\rangle \geq 0} + 2|J'|^2 \geq 0$$

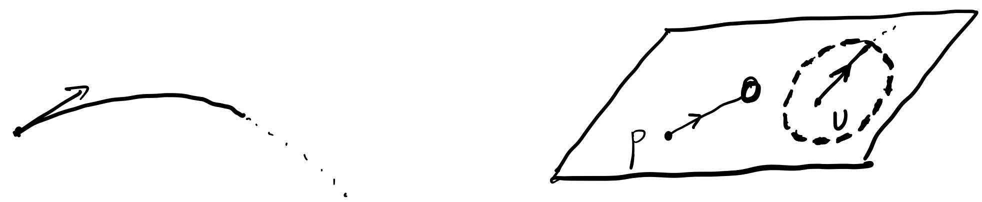
$$\sec(\sigma_{v,w}) = \frac{\langle R(v,w)w, v\rangle}{|v \wedge w|^2} \leq 0 \Rightarrow \langle R(v,w)w, v\rangle \leq 0.$$

$$|v \wedge w|^2 = |v|^2|w|^2 - \langle v, w\rangle^2$$



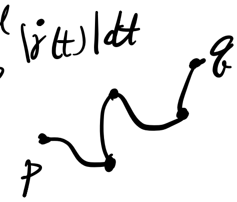
$$f(t) \equiv 0 \Leftrightarrow |J'| \equiv 0 \overset{J(0)=0}{\Rightarrow} J(t) \equiv 0.$$

- **Def**: (M, g) (Riemannian metric) is called <u>geodesically complete</u> if $\forall p \in M$ Exp_p is defined on the whole T_pM.



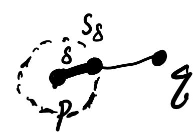
$\big(\forall p \in M, \ Exp_p(tv)$ is defined on $[0, +\infty)$ for any $v \in T_pM\big)$

- $(M, g) \rightsquigarrow$ distance $d(p,q) = \inf\limits_{\substack{\gamma \text{ piecewise diff.} \\ \gamma(0)=p,\ \gamma(\ell)=q}} L(\gamma) = \int_0^\ell |\dot\gamma(t)|dt$



<u>Properties</u>:
- $d(p,q) = d(q,p)$
- $d(p_1, p_2) \leq d(p_1, q) + d(p_2, q)$

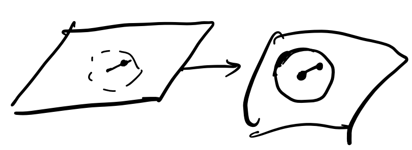
- $d(p,q) \geq 0$, "$=0$" iff $p = q$



$\rightsquigarrow$ metric space (M, d)

$Exp_p(B_\delta(0)) = B_\delta(p)$

$\rightsquigarrow$ <u>metric</u> topology = topology of M

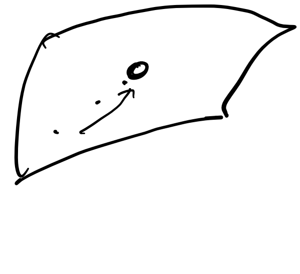


metric space is <u>(metrically) complete</u> if:

$\{p_n\}$ <u>Cauchy sequence</u> $\Longrightarrow \exists\, p_\infty$ s.t. $d(p_n, p_\infty) \to 0$

$p_\infty \in M$

$(\forall \varepsilon > 0, \exists N, \text{ s.t. } d(p_m, p_n) < \varepsilon, \ m, n > N)$



Thm (Hopf-Rinow) The following conditions are equivalent.

1. $\underline{Exp_p : T_pM \to M \text{ is defined for some } p \in M}$.
2. M is geodesically complete ($\forall p \in M$, $Exp_p : T_pM \to M$ is defined)
3. M is metrically complete ($\Longleftrightarrow$ <u>closed and bounded sets are compact</u>)

In addition, any of these condition imply:

4. $\forall p, q \in M$, there exists a minimizing geodesic γ connecting p and q, e.g. $L(\gamma) = d(p, q)$.

<u>Cor</u>: If M is a <u>closed</u> Riem. mfd, then M is complete.
$\qquad\qquad\qquad\qquad\qquad \shortparallel$
(compact without boundary)