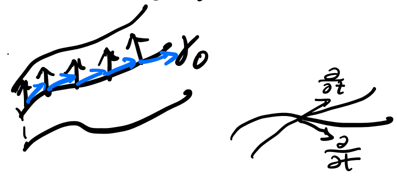
$\gamma : [0,l] \to M$ geodesic : $\nabla_{\dot\gamma}\dot\gamma = 0$

parametrized surface $\boxed{\gamma(s,t) : (-\varepsilon,\varepsilon)\times[0,l] \to M}$

$\gamma_s(t) = \gamma(s,t)$



For fixed s, γ_s is a geodesic

(Geodesic variation)

$$J(t) = \frac{\partial}{\partial s}\gamma(s,t)\Big|_{s=0} \text{ is a Jacobi field.} \qquad \frac{\partial}{\partial t} = \dot\gamma$$

$$\frac{\partial}{\partial s}\gamma(s,t)\Big|_{s=0} = \frac{\partial}{\partial s}$$

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0. \qquad \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} - \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}$$

$$\nabla_{\frac{\partial}{\partial t}}\underline{\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}} = \nabla_{\frac{\partial}{\partial t}}\underline{\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}}$$

$$\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} = \nabla_{\dot\gamma}\nabla_{\dot\gamma}J$$

$$= \underline{\left(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t} - \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} - \nabla_{[\frac{\partial}{\partial t},\frac{\partial}{\partial s}]}\frac{\partial}{\partial t}\right)} + \nabla_{\frac{\partial}{\partial s}}\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right) + \underline{\nabla_{[\frac{\partial}{\partial t},\frac{\partial}{\partial s}]}\frac{\partial}{\partial t}}$$

$$\nabla_{\frac{\partial}{\partial s}}\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right) = \nabla_{\frac{\partial}{\partial s}}\nabla_{\dot\gamma_s}\dot\gamma_s = 0, \qquad \nabla_{[\frac{\partial}{\partial t},\frac{\partial}{\partial s}]}\frac{\partial}{\partial t} = 0$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J = J'' = R\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right)\frac{\partial}{\partial t} = R(\dot\gamma, J)\dot\gamma$$

Jacobi equation: $\boxed{J'' = R(\gamma', J)\gamma'}$

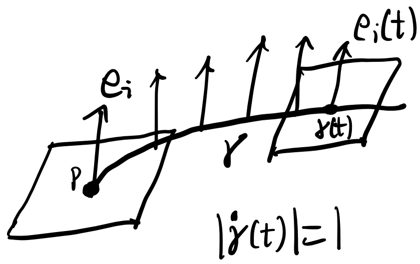
<u>Do Carmo:</u> $\underline{\tilde R(X,Y)Z} = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z = -R(X,Y)Z.$

<u>We use:</u> $\underline{R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z}$

sectional curvature : For each 2-dim. plane $\sigma \subset T_pM$, $\sigma = \operatorname{Span}\{X, Y\}$

$$\sec(\sigma) = \frac{(R(X,Y)Y, X)}{|X|^2|Y|^2 - (X,Y)^2} \overset{\{X,Y\}\text{ o.n.b.}}{=\!=} (R(X,Y)Y, X) = R(X,Y,Y,X)$$

$$\boxed{(R(X,Y)X, Y)}$$



$|\dot\gamma(t)| = 1$

choose o.n.b. $\{e_1, e_2, \cdots, \underset{\gamma'(0)}{\underset{\parallel}{e_n}}\}$ for T_pM

parallel transport to get o.n.b. for $T_{\gamma(t)}M$

$\{e_1(t), e_2(t), \cdots, e_n(t)\}$

(Geodesic γ: $|\dot\gamma(t)| = \text{const}$)

In other words, $\nabla_{\dot\gamma} e_i(t) = 0, \ \forall t \in [0, \ell]$.

$$\nabla_{\underset{a^m(t)\partial_m}{\underset{\parallel}{\dot\gamma}}} \left(\sum_k y^k(t)\partial_k\right) = \dot y^k(t)\partial_k + y^k(t)a^m(t)\cdot\Gamma_{mk}^p\partial_p$$

$$\begin{cases} 0 = \boxed{\left(\dot y^k(t) + y^q(t)a^m(t)\Gamma_{qm}^k(\gamma(t))\right)}\partial_k \\ y^k(0)\partial_k = e_i(0) = e_i \end{cases}$$

Parallel transport: $P_p^{\gamma(t)} : T_pM \to T_{\gamma(t)}M$ (isomorphism of inner product spaces)

$$X \longmapsto X(t)$$
$$Y \longmapsto Y(t)$$

$$\left(\frac{\partial}{\partial t}(X(t), Y(t)) = (\underbrace{\nabla_{\dot\gamma}X}_{=0}, Y) + (X, \underbrace{\nabla_{\dot\gamma}Y}_{=0}) = 0\right)$$

$$\rightsquigarrow \quad J(t) = \sum_{i=1}^n f_i(t)\, e_i(t)$$

$$\nabla_{\dot\gamma}J = J'(t) = \sum_i f_i'(t)\cdot e_i(t) + f_i(t)\nabla_{\dot\gamma}e_i(t) = \sum_i f_i'\cdot e_i$$

$$J''(t) = \sum_i f_i''\cdot e_i = \sum_j f_j''\cdot e_j$$

$$R(\gamma', J)\gamma' = \sum_j \Big(R\big(e_n, \sum_i f_i e_i\big)e_n, e_j\Big) e_j = \sum_{i,j} f_i \big(R(e_n, e_i)e_n, e_j\big)e_j$$

$$= \sum_{i,j} f_i\, R_{ninj}\, e_j$$

$$J'' = R(\gamma', J)\gamma' \iff f_j'' = \sum_{i,j} \underset{\substack{\| \\ a_{ij}(\gamma(t))}}{R_{nimj}} f_i, \quad j = 1, \dots, n.$$

(linear system 2nd order ODE)

<u>Existence & Uniqueness</u>: $\forall\ J(0), J'(0) \in T_pM$, $\exists$ a unique Jacobi field $J(t)$ along $\gamma : [0, l] \to M$.

<u>Special case</u>: Normal Jacobi field: $(J, \gamma') \equiv 0$.

- (Normal) Jacobi fields on mflds of <u>constant sectional curvature</u>

$\forall X, Y, Z, W$, $R(X,Y)Z = k\big((Y,Z)X - (X,Z)Y\big) \iff \big(R(X,Y)Z, W\big) = k\big((X,W)(Y,Z) - (X,Z)(Y,W)\big)$

$\Updownarrow$

$\sec(\sigma) \equiv K$ constant. $\left(\dfrac{(R(X,Y)Y, X)}{(X,X)(Y,Y) - (X,Y)^2} \right)$

$$R(\gamma', J)\gamma' = k\big(\underbrace{(J,\gamma')}_{=0}\gamma' - \underbrace{|\gamma'|^2}_{=1} J\big)$$

$$J'' = -k \cdot J$$

$J = \sum_i f_i e_i(t)$

$$\boxed{J'' + k \cdot J = 0} \iff \underline{f_i'' + k \cdot f_i = 0}, \quad \forall i = 1, \dots, n.$$

$$\Longrightarrow \quad J(t) = \begin{cases} \dfrac{\sin(t\sqrt{k})}{\sqrt{k}} w(t) & k > 0. \\ t \cdot w(t) & k = 0 \\ \dfrac{\sinh(t\sqrt{-k})}{\sqrt{-k}} w(t) & k < 0 \end{cases}$$

($w(t)$ parallel vector field along γ)

$\underline{(\gamma'(0), w(0)) = 0}$ $\Leftrightarrow$ $\boxed{(\gamma'(0), J'(0)) = 0}$

$\Downarrow$

$\underline{(\gamma'(t), w(t)) = 0}$.

$\wr$

$\underline{(\gamma'(t), J(t))}$

Normal Jacobi field satisfying: $\begin{cases} J(0) = 0 \\ J'(0) = w(0). \end{cases}$

Lemma: If $J(t)$ is a Jacobi field along geodesic γ,

Then $(J(t), \gamma'(t)) = (J'(0), \gamma'(0))\,t + (J(0), \gamma'(0))$, $\forall t$

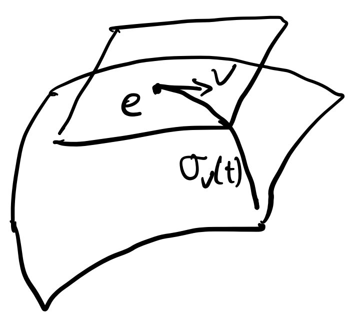
So: J is Normal $\iff (J(0), \gamma'(0)) = 0$ and $(J'(0), \gamma'(0)) = 0$

- $$\begin{aligned} Exp_p : T_pM &\longrightarrow M \\ \rotatebox{90}{$\in$}\quad & \qquad \rotatebox{90}{$\in$} \\ v \quad &\longmapsto \gamma_v(1) \end{aligned}$$

γ_v geodesic satisfies $\begin{cases} \cdot\ \gamma_v(0) = p \\ \cdot\ \gamma_v'(0) = v \end{cases}$

Rmk: Compact Lie gp. G e.g. $G = SO(n)$

$$\begin{aligned} \mathfrak{g} = Lie(G) = T_eG &\xrightarrow{Exp} G \\ \rotatebox{90}{$\in$}\quad & \qquad \rotatebox{90}{$\in$} \\ v \quad &\longmapsto \sigma_v(1) = Exp(v) \end{aligned}$$

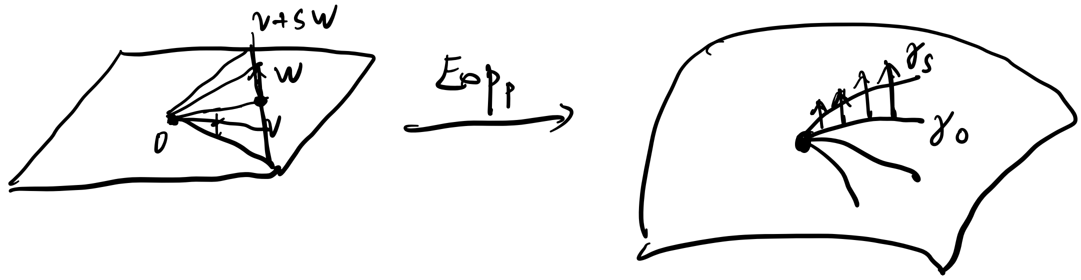


(Fact: $\exists$ G- bi-invariant Riem. metric on G)

$\mathfrak{g} = so(n) = \{A \in M_{n\times n} : A^t + A = 0,\ trA = 0\}$

$$\begin{aligned} Exp : so(n) &\longrightarrow SO(n) \\ \rotatebox{90}{$\in$}\quad & \\ A \quad &\longmapsto Exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} \end{aligned}$$

$$\underline{J(t) = (d\,Exp_p)_{tv}(tw)} = \frac{\partial}{\partial s} \underbrace{Exp_p(t\boxed{v+sw})}_{\gamma(s,t)}\Big|_{s=0}$$



$$(d\,Exp_p)_{tv} : \underset{\underset{T_pM}{\cong}}{T_{tv}(T_pM)} \longrightarrow T_{Exp_p(tv)}M.$$

$\underline{v = \gamma'(0)}$: $\underset{\gamma(t)}{Exp_p(tv)}$

Lem: $\gamma(t): [0, l] \to M$ a geodesic. Then a Jacobi field $\underline{J(t)}$ along γ with $J(0)=0$ is given by

$$\boxed{J(t) = (d\,Exp)_{tv}(tw), \quad \text{where } w = J'(0)}$$

Proof: Just need show $\boxed{(\, J(0)=0, \quad J'(0) = w \,)}$

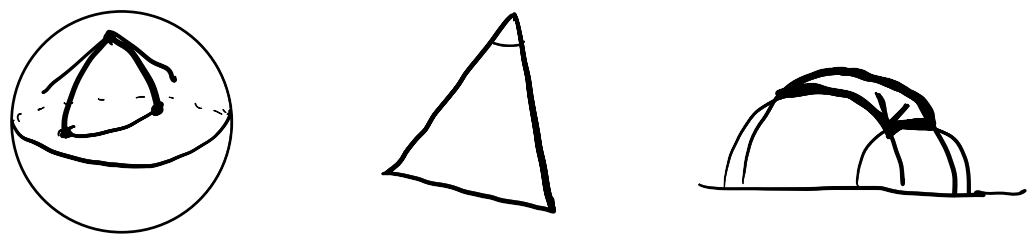
$$J(t) = \frac{\partial}{\partial s} Exp_p(t(\gamma'(0)+sw))$$
$$= (d\,Exp_p)_{t\gamma'(0)}(tw) = t\,\underline{(d\,Exp_p)_{t\gamma'(0)}(w)}$$

$$J'(t) = (d\,Exp_p)_{t\gamma'(0)}(w) + t \cdot \nabla_{\frac{\partial}{\partial t}}(\qquad)$$
$$J'(0) = (d\,Exp_p)_0(w) + 0 = Id_{T_p(M)}(w) = w \qquad \blacksquare$$

• **Prop:** $J(t) = (d\exp_p)_{tv}(tw)$ satisfies near $t=0$:

$|v|=|w|=1$ $\quad |J(t)|^2 = 1 \cdot t^2 - \frac{1}{3}\underline{(R(v,w)w,v)}\, t^4 + \underset{\underset{o(t^4)}{\parallel}}{\underline{R(t)}}$

Rmk: $\sec = k > 0$ $\qquad \sec = 0$ $\qquad \sec = k < 0$



Pf: $\frac{d}{dt}(J,J) = 2(J',J) \xrightarrow{t=0} 0$

$$\frac{d^2}{dt^2}(J,J) = 2(J'',J) + 2(J',J') \xrightarrow{t=0} 2\underset{\underset{2|w|^2=2}{\parallel}}{|J'(0)|^2}$$

$$\frac{d^3}{dt^3}(J,J) = 2(J''',J) + 2(J'',J') + 4(J'',J') = 2(J''',J) + 6(J'',J').$$

$$\left.\frac{d^4}{dt^4}(J,J)\right|_{t=0} = 2(J'''(0), J'(0)) + 6(J'''(0), J'(0))$$

$$J''' = \nabla_{\gamma'}(R(\gamma',J)\gamma') = R(\gamma',J')\gamma' + (\nabla_{\gamma'}R)(\gamma',\underset{\underset{0}{\parallel}}{J})\gamma' \quad t=0$$