M : smooth mfd. TM : tangent bundle

Riem. metric : $g \in C^\infty(Sym^2 T^*M)$

$g_p : T_pM \times T_pM \to \mathbb{R}$ inner product (varying smoothly in $p \in M$).

$(X_1, X_2) \mapsto (X_1, X_2)_g = (X_1, X_2)$

(Levi-Civita) connection : $\nabla = \nabla^{LC} : C^\infty(TM) \to \underline{C^\infty(T^*M \otimes TM)}$

$X \mapsto \nabla X$

$\nabla(f \cdot X) = df \otimes X + f \nabla X$. Leibniz rule.

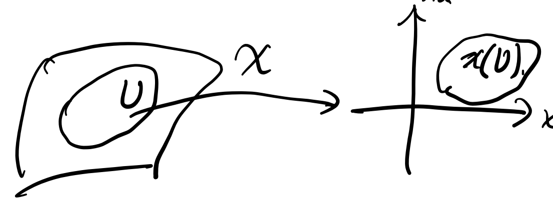
$\langle \nabla X, Y \rangle = \nabla_Y X$ satisfying

(i) $\nabla_X (Y, Z)_g = (\nabla_X Y, Z)_g + (Y, \nabla_X Z)_g$ $(\Leftrightarrow \nabla g = 0)$.

[compatible with the Riem. metric g]. $\forall X, Y, Z \in C^\infty(TM)$.

(ii) (Torsion free) $\nabla_X Y - \nabla_Y X = [X, Y]$ $\forall X, Y \in C^\infty(TM)$

In local coordinates. $(U, \{x_i\}_{i=1}^n)$, $U \subset M^n$



$g = \sum_{i,j=1}^n g_{ij}\, dx^i \otimes dx^j$ $\quad (g_{ij})_{n \times n} > 0$ $\quad g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $\frac{\partial}{\partial x_i} = \partial_i$, $\frac{\partial}{\partial x_j} = \partial_j$.

$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$ $\quad \Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

$\nabla_{\partial_i} \partial_j = \langle \nabla \partial_j, \partial_i \rangle$ $\quad \Gamma_{ij}^k$: Christoffel symbol $\quad \Gamma_{ij}^k = \Gamma_{ji}^k$ ($\leftrightarrow$ Torsion free).

Curvature: $X, Y, Z \in TM$, $\boxed{R(X,Y): T_pM \to T_pM}$

$$\underline{R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z \in TM.}$$

$$[X,Y] = \nabla_X Y - \nabla_Y X.$$

$$= \underline{(\nabla_X\nabla_Y Z - \nabla_{\nabla_X Y} Z)} - \underline{(\nabla_Y\nabla_X Z - \nabla_{\nabla_Y X} Z)}$$

- $R(\underline{X,Y})Z = -R(Y,X)Z$.
- $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$

$\Big\} \Rightarrow R \in C^\infty(\Lambda^2 T^*M \otimes \underline{o(TM,g)})$

$$\langle R(X,Y)Z, W\rangle = R(X,Y,Z,W)$$

$$R \in C^\infty(\mathrm{Sym}^2(\Lambda^2 TM)).$$

- $R(X,Y,Z,W) = R(Z,W,X,Y)$

- <u>1st. Bianchi identity</u>

$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0.$$

- 2nd. Bianchi identity $\boxed{d^\nabla R = 0}$

- Geodesics: $\gamma: [0, l] \to M$ satisfies
  $t \in [0,l]$

$$\nabla^{LC}_{\dot\gamma}\dot\gamma = 0 \iff \boxed{\ddot{x}^j - \dot{x}^i\dot{x}^k\Gamma^j_{ik}(x(t)) = 0,\ 1 \le j \le n.}$$

$$\dot\gamma = \frac{d}{dt}\gamma(t) = \sum_i \dot{x}^i \cdot \partial_i. \qquad \nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot{x}^i\partial_i}(\dot{x}^j\partial_j) = \ddot{x}^j\partial_j - \dot{x}^i\dot{x}^k\Gamma^j_{ik}\partial_j$$

$$= (\ddot{x}^j - \dot{x}^i\dot{x}^k\Gamma^j_{ik})\partial_j$$

$$\gamma(t) = (x_1(t), \cdots, x_n(t))$$

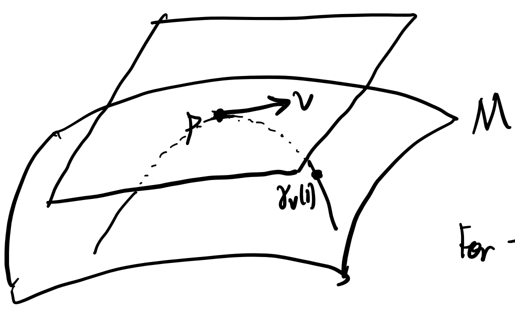
Local Existence and Uniqueness for geodesics:
(for ODE)

$M \ni p$. $v \in T_pM$

local
$\exists$ unique Geodesic $\gamma : \underline{(-\varepsilon, \varepsilon)} \to M$
$\gamma(0) = p$, $\dot{\gamma}(0) = v$.

- Exponential map: $\mathrm{Exp}_p : \begin{array}{c} v \\ \Upsilon \\ U_p \\ \cap \\ T_pM \end{array} \begin{array}{c} \longmapsto \\ \\ \longrightarrow \\ \\ \\ \end{array} \begin{array}{c} \gamma_v(1) \\ \Upsilon \\ M \\ \\ \\ \end{array}$

$\Downarrow$ (sm. dependence on initial data)



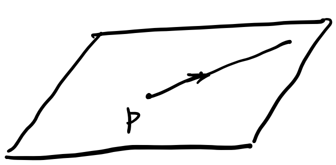
$\mathrm{Exp} : \begin{array}{c} U \\ \cap \\ TM \end{array} \longrightarrow M$ smooth map

For fixed $p \in M$, $d\mathrm{Exp}_p(0) = (\mathrm{Id}_p : T_pM \to T_pM)$

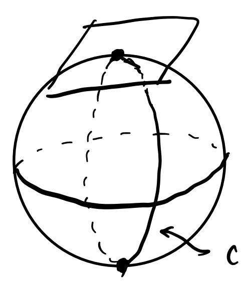
$\Rightarrow$ Exp_p is a local diffeomorphism.

<u>Fact</u>: If M is closed, then $\mathrm{Exp} : TM \to M$ is well defined.
(compact without boundary)

<u>Ex</u>: $\underline{\mathbb{R}^n}$



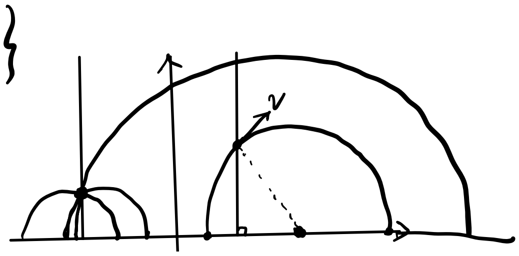
$\mathbb{S}^n = \{ y \in \mathbb{R}^{n+1} : |y| = 1 \}$



closed geodesics

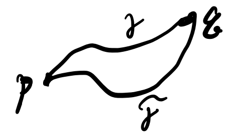
$\underline{\mathbb{H}^n} \rightsquigarrow \mathbb{H}^2 = \{ \underset{x+iy}{\overset{\shortparallel}{z}} \in \mathbb{C} : \mathrm{Im}\, z > 0 \}$

$g = \dfrac{dx^2 + dy^2}{y^2}$



- $d(p,q) = \inf\limits_{\substack{\gamma:\ \text{piecewise diff.} \\ \gamma(0)=p,\ \gamma(t_1)=q}} L(\gamma)$ $\qquad L(\gamma) = \int_0^{t_1} |\dot\gamma|_g \, dt$

$p, q \in M$

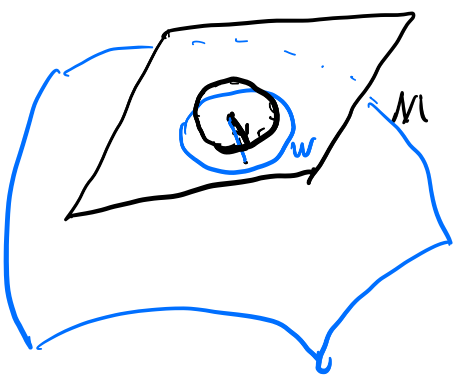


<u>Fact</u>: · Geodesics are locally length minimizing.

$$\begin{aligned} Exp_p : T_pM &\longrightarrow M \\ v &\longmapsto \gamma_v(1) = \gamma_{\frac{v}{|v|}}(|v|) \end{aligned}$$

Assume $B_\delta(0) \subset T_pM$ s.t. $B_\delta(0) = \{v : |v| < \delta\}$

$Exp_p|_{B_\delta(0)} : B_\delta(0) \longrightarrow W \subset M$ diffeomorphism.

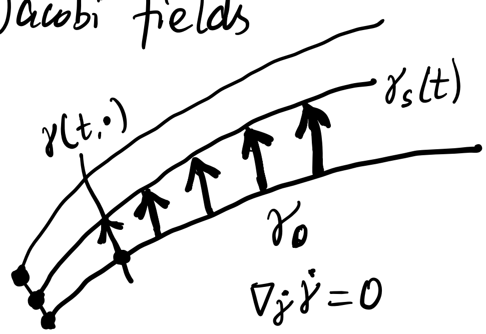


Then $\underset{W}{q} = Exp_p(\underset{B_\delta(0)}{v_q})$

$\gamma_q(t) = Exp_p(tv_q)$ satisfies

$L(\gamma_q(t)) = d(p, \gamma_q(t))$, $0 \le t \le 1$.

- Jacobi fields



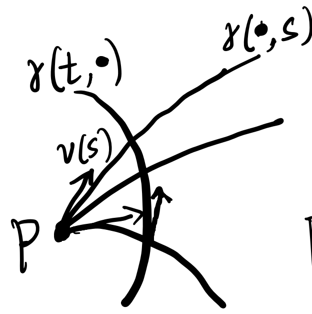
(geodesic variation)

$\gamma : [0,\ell) \times (-\varepsilon, \varepsilon) \to M$

$\gamma_s(t) = \gamma(t,s)$ geodesics.

$\nabla_{\dot\gamma_s} \dot\gamma_s = 0$. $\quad \dot\gamma_s = \dfrac{\partial\gamma}{\partial t}$

$$J(t) = \frac{\partial}{\partial s}\gamma(t,s)\Big|_{s=0} \qquad \{v(s)\} \in T_pM$$



$$\gamma(t,s) = \overset{\phi}{\mathrm{Exp}_p}\big(\overset{\sigma(s)}{t\cdot v(s)}\big)$$

$$\boxed{J(t) = \frac{\partial}{\partial s}\gamma(t,s)\Big|_{s=0} = (d\,\mathrm{Exp}_p)_{tv(0)}(tv'(0))}$$

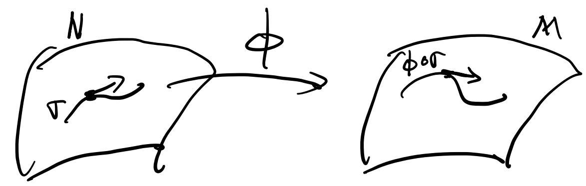
$J(0)=0$

$\mathrm{Exp}_p : T_pM \to M$ smooth

$\phi : \underset{\substack{\cup \\ z}}{N} \to M$ sm. map

$\phi_{*z} = (d\phi)_z : T_zN \to T_{\phi(z)}M$

$z = \sigma(0)$ $\qquad \underset{\substack{\parallel \\ \frac{d}{ds}\sigma(s)|_{s=0}}}{\dot\sigma(0)} \mapsto \frac{d}{ds}\phi(\sigma(s))\Big|_{s=0}$



Prop. Any Jacobi field $J(t)$ along a geodesic $\gamma : [0,\ell] \to M$ with $J(0)=0$ is of the form

$$\boxed{(d\,\mathrm{Exp}_p)_{t\dot\gamma(0)}(tw)} \quad \text{for some } w \in T_pM.$$

$(\mathrm{Exp}_p)_{*(t\dot\gamma)} = (d\,\mathrm{Exp}_p)_{t\dot\gamma(0)} : T_pM \to T_{\gamma(t)}M.$