!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(20pts) (1): Write down the Cauchy-Riemann equation.
(2): Suppose $f(z) = u + iv$ is holomorphic and $u(x, y) = x^3 + cxy^2$ with $c \in \mathbb{C}$. Find the complex number c and calculate the function $v = \mathrm{Im}(f)$.

(1) $f = u + iv$ is holomorphic $\Longleftrightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$ (Cauchy-Riemann equation) 5

(2) $u = \mathrm{Re}(f)$ is harmonic $\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + c \cdot y^2) = 6x, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(cx \cdot 2y) = 2c \cdot x$

$\Rightarrow 0 = 6x + 2c \cdot x \Rightarrow 2c + 6 = 0 \Rightarrow c = -3.$ 5

By Cauchy-Riemann equation, $\left\{ \begin{array}{l} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = cx \cdot 2y = 6xy \Rightarrow v(x, y) = 3x^2y + g(y) \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \end{array} \right\}$

$\Rightarrow 3x^2 - 3y^2 = \frac{\partial v}{\partial y} = 3x^2 + g'(y) \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + \text{const}.$ 5

choose const=0.

$\Rightarrow v(x, y) = 3x^2y - y^3$ 5

$\Rightarrow f(z) = u + iv = (x^3 - 3x \cdot y^2) + i(3x^2y - y^3) = (x + iy)^3 = z^3.$

2(20pts) Let $f(z)$ be a holomorphic function defined on the punctured plane $\mathbb{C}\setminus\{0\}$ whose image set is contained in the region outside of the unit disk: $f(\mathbb{C}\setminus\{0\}) \subseteq \{w : |w| > 1\}$. Prove that $f(z)$ is constant function.

Proof: Consider $g(z) = \frac{1}{f(z)}$. Then g is holomorphic on $\mathbb{C}\setminus\{0\}$ 5

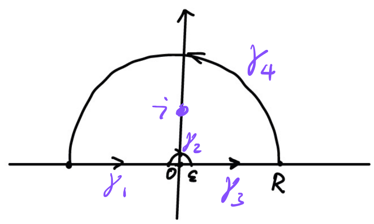
and $|g(z)| = \frac{1}{|f(z)|} \leq 1$. So 0 is a removable singularity and 5

$g(z)$ extends to a holomorphic function on $\mathbb{C}$ that is uniformly bounded.

By Liouville Thm, $g(z)$ is a constant function that can't be 0. 5

So $f(z) = \frac{1}{g(z)}$ is also a nonzero constant function. 5

3(20 pts) Calculate integral: $\displaystyle\int_{-\infty}^{+\infty}\frac{\sin(x)}{x(x^2+1)^2}dx$. Hint: use the contour:



5

Consider the contour integral $\int_\gamma f(z)\,dz$ with $f(z)=\frac{e^{iz}}{z(z^2+1)^2} = \frac{e^{iz}}{z(z+i)^2(z-i)^2}$

$f(z)$ has a pole at i of order 2. Its residue:

$$\mathrm{Res}_{z=i} f(z) = \left(\frac{d}{dz}(z-i)^2 f(z)\right)\Big|_{z=i} = \frac{d}{dz}\frac{e^{iz}}{z(z+i)^2}\Big|_{z=i} = \left(\frac{e^{iz}\cdot i}{z(z+i)^2} - \frac{e^{iz}}{z^2(z+i)^2} - \frac{2\cdot e^{iz}}{z(z+i)^3}\right)\Big|_{z=i}$$

$$= \frac{e^{-1}\cdot i}{i\cdot(-4)} - \frac{e^{-1}}{(-1)(-4)} - \frac{2\cdot e^{-1}}{i\cdot 8\cdot(-i)} = \frac{1}{4}e^{-1}\cdot(-1-1-1) = -\frac{3}{4}e^{-1}.$$ 5

- $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{-R}^{-\varepsilon}\frac{\overset{\cos x + i\cdot\sin x}{\overset{\shortparallel}{e^{ix}}}}{x(x^2+1)^2}dx = \int_R^{\varepsilon}\frac{e^{-ix}}{x(x^2+1)^2}dx = -\int_{\varepsilon}^{R}\frac{e^{-ix}}{x(x^2+1)^2}dx$

- $\displaystyle\int_{\gamma_1+\gamma_3} f(z)\,dz = \int_{\varepsilon}^{R}\frac{e^{ix}-e^{-ix}}{x(x^2+1)^2}dx = 2i\cdot\int_{\varepsilon}^{R}\frac{\sin x}{x(x^2+1)^2}dx \xrightarrow[R\to\infty]{\varepsilon\to0} 2i\int_0^{\infty}\frac{\sin x}{x(x^2+1)}dx = i\int_{-\infty}^{+\infty}\frac{\sin x}{x(x^2+1)^2}dx$

- $\displaystyle\int_{\gamma_2} f(z)\,dz = \int_{\pi}^{0}\frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}(1+\varepsilon^2e^{2i\theta})}\varepsilon e^{i\theta} i\,d\theta = i\int_{\pi}^{0}\Big(\underbrace{\frac{e^{i\varepsilon e^{i\theta}}}{(1+\varepsilon^2e^{2i\theta})^2}}_{1+O(\varepsilon)}\Big)d\theta \xrightarrow{\varepsilon\to0} i\int_{\pi}^{0}d\theta = -i\pi.$ 5

  γ_2: $z=\varepsilon\cdot e^{i\theta}$, $\theta: \pi\to0$
  $dz=\varepsilon e^{i\theta}\cdot i\,d\theta$

  $$\left(\left|\frac{e^{i\varepsilon e^{i\theta}}}{(1+\varepsilon^2e^{2i\theta})^2}-1\right| \le C\cdot\varepsilon \xrightarrow{\varepsilon\to0} 0 \text{ uniformly}\right)$$

- $\displaystyle\left|\int_{\gamma_4} f(z)\,dz\right| = \left|\int_0^{\pi}\frac{e^{iRe^{i\theta}}}{\cancel{Re^{i\theta}}(1+R^2e^{2i\theta})^2}\cancel{Re^{i\theta}}\,i\,d\theta\right| \le \int_0^{\pi}\frac{1}{(R^2-1)^2}d\theta = \frac{2\pi}{(R^2-1)^2}\xrightarrow{R\to+\infty}0$

γ_4: $z=R\cdot e^{i\theta}$, $0\le\theta\le\pi$. $\quad |e^{iRe^{i\theta}}| = |e^{i(R\cos\theta+iR\sin\theta)}| = |e^{-R\sin\theta}\cdot e^{iR\cos\theta}| = e^{-R\sin\theta}\le 1$

$$\Rightarrow\ i\int_{-\infty}^{+\infty}\frac{\sin x}{x(1+x^2)^2}dx - i\cdot\pi = 2\pi i\cdot\left(-\frac{3}{4}e^{-1}\right)$$

5

$$\Rightarrow\ \int_{-\infty}^{+\infty}\frac{\sin x}{x(1+x^2)^2}dx = \pi - \frac{3}{2}\pi e^{-1} = \frac{\pi}{2}\left(2-\frac{3}{e}\right).$$

4(20pts) Use Rouché's theorem to find the number of zeros of the function $f(z) = z^4 - 4z^2 + z + 1$ in the annulus $1 < |z| < 3$.

· At $|z|=1$, $|z^4+z+1| \leq |z|^4+|z|+1 = 3 < |-4z^2|$

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$\Rightarrow$ z^4-4z^2+z+1 and $-4z^2$ have the same number of roots inside $|z|\leq 1$. The number is 2.

· At $|z|=3$, $|-4z^2+z+1| \leq 4|z|^2+|z|+1 = 4\cdot 3^2+3+1 = 40 < |z|^4 = 81$

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$\Rightarrow$ z^4-4z^2+z+1 and z^4 have the same number of roots inside $|z|\leq 3$. The number is 4.

So there are $4-2=2$ roots in the annulus $1<|z|<3$.

5(20pts) Find the Taylor series of the following function centered at 0. What is the radius of convergence? (Hint: use partial fractions and the fact that series can be differentiated term by term).

$$f(z) = \frac{z}{(z-2)^2}$$

$$\frac{1}{z-2} = -\frac{1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} 2^{-(n+1)} z^n .$$ 5

$$\Rightarrow \frac{1}{(z-2)^2} = -\frac{d}{dz}\left(\frac{1}{z-2}\right) = -\frac{d}{dz}\left(-\sum_{n=0}^{\infty} 2^{-(n+1)} z^n\right) = \sum_{n=1}^{+\infty} 2^{-(n+1)} z^{n-1}$$ 5

$$\Rightarrow \frac{z}{(z-2)^2} = \sum_{n=1}^{\infty} n \cdot 2^{-(n+1)} z^n .$$ 5

$$a_n = n \cdot 2^{-(n+1)} \Rightarrow \lim_{n\to+\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to+\infty} n^{\frac{1}{n}} 2^{-\frac{n+1}{n}} = 2^{-1} \cdot \lim_{n\to+\infty} n^{\frac{1}{n}} \cdot \lim_{n\to+\infty} 2^{-\frac{1}{n}}$$

$$= 2^{-1} \cdot 1 \cdot 1 = 2^{-1}$$ 5

$\Rightarrow$ radius of convergence $= \dfrac{1}{\frac{1}{2}} = 2.$

OR: $$\frac{z}{(z-2)^2} = \frac{z-2+2}{(z-2)^2} = \frac{1}{z-2} + \frac{2}{(z-2)^2}. \qquad \frac{1}{z-2} = -\frac{1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{+\infty} \frac{z^n}{2^n}$$

$$\Rightarrow \frac{2}{(z-2)^2} = -2 \cdot \frac{d}{dz}\frac{1}{(z-2)} = -2 \cdot \left(-\frac{1}{2}\right)\sum_{n=1}^{+\infty} \frac{n \cdot z^{n-1}}{2^n} = \sum_{n=0}^{+\infty} \frac{(n+1)\cdot z^n}{2^{n+1}}$$

$$\Rightarrow \frac{z}{(z-2)^2} = -\sum_{n=0}^{+\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{+\infty}\left(\frac{n}{2^{n+1}} + \frac{1}{2^{n+1}}\right) z^n = \sum_{n=0}^{+\infty} \frac{n}{2^{n+1}} z^n .$$

$$\Rightarrow a_n = \frac{n}{2^{n+1}}, \quad \lim_{n\to+\infty} a_n^{1/n} = \lim_{n\to+\infty} \frac{n^{\frac{1}{n}}}{2^{\frac{n+1}{n}}} = \frac{1}{2} \Rightarrow \text{radius of convergence} = 2.$$