!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(10pts) Assume $f(z) = \log z$ is the branch that equals the standard natural logarithmic function for positive z, defined away from the negative real axis.

(1) Calculate the Taylor expansion of $f(z)$ centered at $z = 1$. What is its radius of convergence?

(2) What is the singularity type of the meromorphic function $f(z)/(z-1)$ at $z = 1$? Explain your reason.

(1) $f'(z) = \frac{1}{z},\ f''(z) = -\frac{1}{z^2},\ f^{(3)}(z) = \frac{2}{z^3}, \cdots, f^{(n)}(z) = (-1)^{n-1}\cdot\frac{(n-1)!}{z^n}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{(n-1)!}{n!}(z-1)^n = \sum_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{1}{n}(z-1)^n.$$

$a_n = (-1)^{n-1}\cdot\frac{1}{n}$. $\lim_{n\to+\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to+\infty} n^{-\frac{1}{n}} = e^0 = 1.$

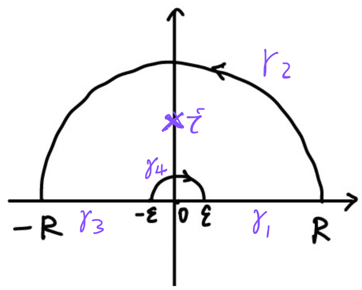
$n^{-\frac{1}{n}} = e^{-\frac{1}{n}\log n}$

$\Rightarrow$ radius of convergence $= \dfrac{1}{\lim_{n\to+\infty}|a_n|^{1/n}} = 1$.

(2) $\lim_{z\to 1} \frac{f(z)}{z-1} = \lim_{z\to 1} \frac{\log z - \log 1}{z-1} = f'(1) = 0$

$\Rightarrow$ $z=1$ is a removable singularity of $\frac{f(z)}{z-1}$.

2(10pts) Calculate the integral $\int_0^{+\infty} \frac{\log x}{(x^2+1)^2} dx$ using the following contour:



$$f(z) = \frac{\log z}{(z^2+1)^2}, \qquad \int_\gamma f(z)dz = 2\pi i \cdot \mathrm{res}_{z=i} f(z).$$

$$\mathrm{res}_{z=i} f(z) = \frac{d}{dz}(z-i)^2 \frac{\log z}{(z^2+1)^2}\Big|_{z=i} = \frac{d}{dz}\frac{\log z}{(z+i)^2}\Big|_{z=i} = \left(\frac{1}{z}\cdot\frac{1}{(z+i)^2} - \log z \cdot 2 \cdot \frac{1}{(z+i)^3}\right)\Big|_{z=i}$$

$$= \frac{1}{i}\cdot\frac{1}{(2i)^2} - \log i \cdot \frac{2}{(2i)^3} = \frac{i}{4} - \frac{\pi}{2}i \cdot \frac{2}{-8i} = \frac{i}{4} + \frac{\pi}{8}$$

$$\int_{\gamma_1} f(z)dz = \int_\varepsilon^R \frac{\log x}{(x^2+1)^2}dx \xrightarrow[R\to+\infty]{\varepsilon\to 0} \int_0^{+\infty} \frac{\log x}{(x^2+1)^2}dx = I$$

$$\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^\pi \frac{\log R + i\theta}{(R^2e^{2i\theta}+1)^2}\cdot Re^{i\theta} i\, d\theta\right| \le \int_0^\pi \frac{\log R+\theta}{(R^2-1)^2} R\, d\theta \xrightarrow{R\to+\infty} 0.$$

$$\int_{\gamma_3} f(z)dz = \int_{-R}^{-\varepsilon} \frac{\log|x| + i\pi}{(x^2+1)^2}dx = -\int_R^\varepsilon \frac{\log x + i\pi}{(x^2+1)^2}dx = \int_\varepsilon^R \frac{\log x + i\pi}{(x^2+1)^2}dx \xrightarrow[R\to+\infty]{\varepsilon\to 0} I + i\pi\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$\left|\int_{\gamma_4} f(z)dz\right| = \left|\int_\pi^0 \frac{\log\varepsilon + i\theta}{(\varepsilon^2e^{2i\theta}+1)^2}\varepsilon e^{i\theta} i\, d\theta\right| \le \int_0^\pi \frac{|\log\varepsilon| + \theta}{(1-\varepsilon^2)^2}\varepsilon\, d\theta \xrightarrow{\varepsilon\to 0} 0.$$

So. $$2\cdot I + i\cdot\pi\int_0^\infty \frac{dx}{(x^2+1)^2} = 2\pi i\cdot\left(\frac{i}{4}+\frac{\pi}{8}\right) = -\frac{\pi}{2} + \frac{\pi^2}{4}i$$

$$\Rightarrow I = \int_0^\infty \frac{\log x}{(x^2+1)^2}dx = -\frac{\pi}{4}, \qquad \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

3(15 pts) Let $f : \mathbb{H} \to \mathbb{D}$ be a holomorphic function that satisfies $f(i) = 0$. Prove the inequality $|f'(i)| \le \frac{1}{2}$. Find the expression of $f(z)$ when $f'(i) = 1/2$.

Consider the composition: $\mathbb{D} \xrightarrow{i\cdot\frac{1-z}{1+z} = w} \mathbb{H} \xrightarrow{f} \mathbb{H}$

$g(z) = f\left(i\frac{1-z}{1+z}\right) : \mathbb{D} \to \mathbb{D}$ satisfies $g(0) = f(i) = 0$.

$$w'(z) = i\cdot\frac{(-1)\cdot(1+z) - (1-z)\cdot 1}{(1+z)^2} = i\cdot\frac{-2}{(1+z)^2}$$

By Schwarz lemma, $|g'(0)| \le 1$

By the chain rule, $g'(z) = f'(w(z))\cdot w'(z) = f'(w(z))\cdot\frac{-2i}{(1+z)^2}$.

So $g'(0) = f'(w(0))\cdot(-2i) = f'(i)\cdot(-2i)$ satisfies

$$|f'(i)\cdot(-2i)| = |f'(i)|\cdot 2 \le 1 \Rightarrow |f'(i)| \le \frac{1}{2}.$$

If $f'(i) = \frac{1}{2}$, then $g'(0) = f'(i)\cdot(-2i) = -i \Rightarrow g(z) = -i\cdot z$

$\Leftrightarrow f\left(i\cdot\frac{1-z}{1+z}\right) = -i\cdot z$. Set $w = i\cdot\frac{1-z}{1+z} \Rightarrow z = \frac{i-w}{i+w}$

$\Updownarrow$

$w + w\cdot z = i - iz \Leftrightarrow z\cdot(w+i) = i - w$

$\Rightarrow f(w) = -i\cdot\frac{i-w}{i+w} = \frac{i\cdot w + 1}{i+w}$

Check: $f'(i) = \frac{i\cdot(i+i) - (i^2+1)\cdot 1}{(2i)^2} = \frac{-2}{-4} = \frac{1}{2}$ ✓

4(10pts) For any integer $n > 0$, consider the polynomial

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}.$$

Prove that for any $R > 0$, there exists $N > 0$ such that when $n \geq N$, $P_n(z)$ does not have any zeros inside the disc $\{|z| < R\}$ (Hint: use Rouché's Theorem to compare $P_n(z)$ with e^z).

Proof: • For fixed R, because $e^z \neq 0$, on $\mathbb{C}$,

$\exists\ \delta > 0$ s.t. $\min\limits_{z \in \partial D_R} |e^z| \geq \delta > 0$. $\left(|e^{Re^{i\theta}}| = |e^{R\cos\theta + i\sin\theta}| = e^{R\cos\theta} \geq e^{-R} = \delta \right)$

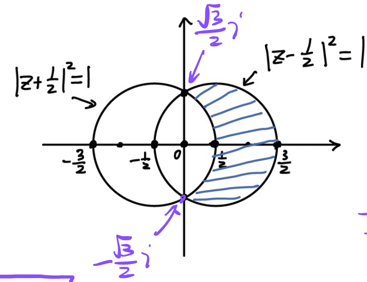
• As $n \to +\infty$, $P_n(z)$ converges uniformly to e^z on $\overline{D_R}$

$\Rightarrow \exists\ N > 0$ s.t. when $n \geq N$, $|P_n(z) - e^z| < \delta \leq |e^z|$ on ∂D_R

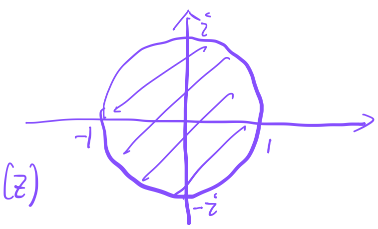
• By Rouché's Theorem, e^z and $P_n(z)$ have the same number of zeros on D_R, which is none since e^z has no zeros.

5(15pts) Find a conformal equivalence map from the following shaded open domain to the unit disc. (Hint: first use a linear fractional transformation)

$$f_1\left(\tfrac{3}{2}\right) = \frac{\frac{3}{2}-\frac{\sqrt{3}}{2}i}{\frac{3}{2}+\frac{\sqrt{3}}{2}i} = \frac{\sqrt{3}-i}{\sqrt{3}+i} = \frac{(\sqrt{3}-i)^2}{4} = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2}-\frac{\sqrt{3}}{2}i$$



$f_3\circ f_2\circ f_1(z)$



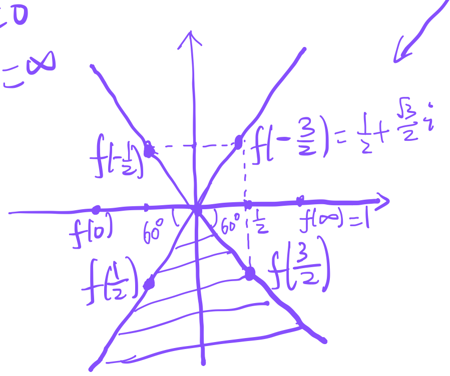
$f(0)=-1$

$f(\frac{\sqrt{3}}{2}i)=0$

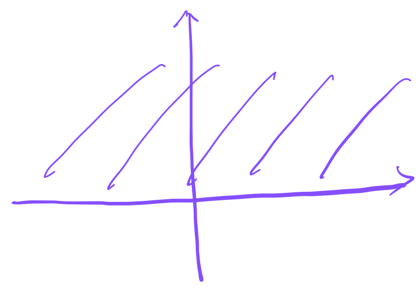
$f(-\frac{\sqrt{3}}{2}i)=\infty$

$$\boxed{z_1 = f_1(z) = \frac{z-\frac{\sqrt{3}}{2}i}{z+\frac{\sqrt{3}}{2}i}}$$

$$\boxed{z_3 = \frac{i-z_2}{i+z_2} = f_3(z_2)}$$



$$\boxed{f_2(z_1) = z_2 = z_1^3}$$



$$f_1\left(\tfrac{1}{2}\right) = \frac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \frac{(1-\sqrt{3}i)^2}{4} = \frac{-2-2\sqrt{3}i}{4} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$f_1\left(-\tfrac{1}{2}\right) = \frac{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}{-\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \frac{1+\sqrt{3}i}{1-\sqrt{3}i} = \frac{-2+2\sqrt{3}i}{4} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$f_1\left(-\tfrac{3}{2}\right) = \frac{-\frac{3}{2}-\frac{\sqrt{3}}{2}i}{-\frac{3}{2}+\frac{\sqrt{3}}{2}i} = \frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{2+2\sqrt{3}i}{4} = \frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$f_1(1) = \frac{1-\frac{\sqrt{3}}{2}i}{1+\frac{\sqrt{3}}{2}i} = \frac{(2-\sqrt{3}i)^2}{|2+\sqrt{3}i|^2} = \frac{1-4\sqrt{3}i}{7}$$

$f(z) = f_3\circ f_2\circ f_1(z)$ is a wanted conformal equivalence to the unit disk.

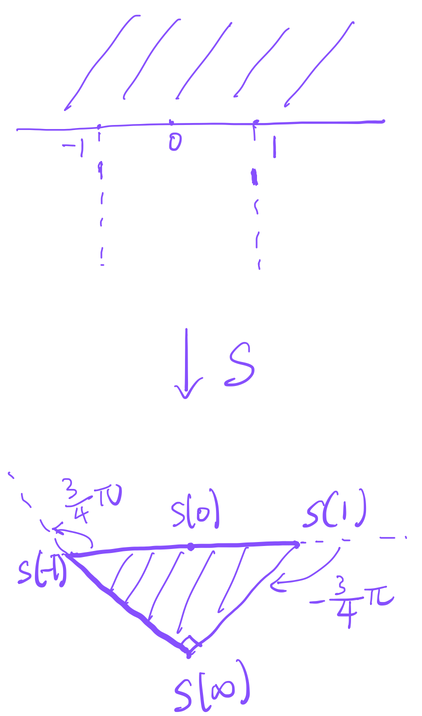
6(10pts) Consider the following Christoffel-Schwarz integral defined on $\mathbb{H}$:

$$S(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^{3/4}}.$$

What kind of polygon is the image $S(\mathbb{H})$ (how many sides does it have and what are the angles)? Prove that the perimeter (total length of sides) of this polygon is $\frac{1+\sqrt{2}}{\sqrt{2\pi}}\Gamma(1/4)^2$.

Choose a branch of $(1-\zeta^2)^{3/4}$ s.t. it is positive when $\zeta \in [-1, 1]$.

Then $$(1-\zeta^2)^{3/4} = (1-\zeta)^{3/4}\cdot(1+\zeta)^{3/4} = \begin{cases} (1-\zeta)^{\frac34}\cdot(-\zeta-1)^{\frac34} e^{i\cdot\pi\frac34} & \zeta<-1 \\ (1-\zeta^2)^{\frac34} & -1<\zeta<1 \\ (\zeta-1)^{\frac34}(1+\zeta)^{\frac34}e^{-i\cdot\pi\frac34} & \zeta>1 \end{cases}$$



Side length: $$|S(-1)S(1)| = \int_{-1}^{1} \frac{dx}{(1-x^2)^{\frac34}} = 2\cdot\int_0^1 \frac{dx}{(1-x^2)^{3/4}}$$

$$\overset{x=t^{\frac12}}{=} 2\cdot\int_0^1 \frac{\frac12 t^{-\frac12}\,dt}{(1-t)^{\frac34}} = \int_0^1 t^{-\frac12}(1-t)^{-\frac34}dt$$

$$= B\left(\tfrac12,\tfrac14\right) = \frac{\Gamma(\frac12)\Gamma(\frac14)}{\Gamma(\frac34)} = \frac{\sqrt{\pi}\cdot\Gamma(\frac14)}{\frac{\pi}{\sin(\frac{\pi}{4})}\cdot\frac{1}{\Gamma(\frac14)}}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt2}{2}\cdot\Gamma\left(\tfrac14\right)^2$$

$$\Rightarrow \text{perimeter} = \left(2\cdot\tfrac{\sqrt2}{2}+1\right)\cdot|S(-1)\cdot S(1)|$$

$$= (\sqrt2+1)\frac{1}{\sqrt{2\pi}}\Gamma\left(\tfrac14\right)^2.$$

$S(\mathbb{H})$ = right triangle with angles 45°, 45° and 90°.

OR: Side $$|S(1)S(\infty)| = \int_1^\infty \frac{dx}{(x^2-1)^{3/4}}dx \overset{x=t^{-\frac12}}{=} \int_1^0 (t^{-1}-1)^{-\frac34}\cdot\left(-\tfrac12\right)\cdot t^{-\frac32}dt = \frac12\int_0^1 t^{-\frac34}(1-t)^{-\frac34}dt$$

$$= \frac12 B\left(\tfrac14,\tfrac14\right) = \frac12\frac{\Gamma(\frac14)^2}{\Gamma(\frac12)} = \frac{\Gamma(\frac14)^2}{2\sqrt{\pi}}$$

$$\Rightarrow \text{perimeter} = (2+\sqrt2)\cdot|S(1)S(\infty)| = (2+\sqrt2)\cdot\frac{\Gamma(\frac14)^2}{2\sqrt\pi} = (\sqrt2+1)\cdot\frac{\Gamma(\frac14)^2}{\sqrt{2\pi}}.$$

7(10pts) Let $f(z) = e^z - 2z$.

(1) Let $f = Ae^{Bz}\prod_{n=1}^{+\infty}(1-\frac{z}{a_n})e^{z/a_n}$ be its Hadamard factorization where $\{a_n\}_{n\geq 1}$ are the zeros of f. Calculate A and B (Hint: consider $f(0)$ and $f'(0)/f(0)$).

(2) Prove that $f(z)$ takes any complex value infinitely many times.

(1) $$\underset{\substack{\| \\ e^0 - 2\cdot 0 = 1}}{f(0)} = Ae^{B\cdot 0}\prod_{n=1}^{+\infty}(1-\frac{0}{a_n})e^{\frac{0}{a_n}} = A \Rightarrow A = 1.$$

$$\underset{\substack{\| \\ \frac{1}{e^z-2z}\cdot(e^z-2)}}{\frac{f'(z)}{f(z)}} = B + \sum_{n=1}^{+\infty}\left(\frac{-\frac{1}{a_n}}{1-\frac{z}{a_n}} + \frac{1}{a_n}\right) \Rightarrow \underset{\substack{\| \\ \frac{1}{e^0-2\cdot 0}\cdot(e^0-2) = \frac{1}{1}\cdot(-1)}}{\frac{f'(0)}{f(0)}} = B \Rightarrow B = -1.$$

(2) For any fixed $w \in \mathbb{C}$, $\underset{\substack{g(z) \\ \|}}{f(z)-w} = e^z - 2z - w$ is entire of growth order 1

$\Rightarrow$ Hadamard factorization: $e^z - 2z - w = C\cdot e^{D\cdot z}\cdot z^m \cdot \prod_{n=1}^{\infty}(1-\frac{z}{b_n})e^{\frac{z}{b_n}}$

where $\{b_n\}$ are zeros of $g(z)$ and $m=1$ if $w=1$ and $m=0$ if $w \neq 1$.

Suppose $f(z)$ takes the value w finitely many times. Then $\{b_n\}$ is a finite set.

So $g(z) = e^z - 2z - w = e^{Dz}\cdot P(z)$ for a polynomial $P(z)$.

By taking the 2nd order derivative with respect to z, we get

$e^z = e^{Dz}\cdot Q(z)$ for a polynomial Q. Then we must have $D = 1$.

$\Rightarrow e^z - 2z - w = e^z\cdot P(z) \Rightarrow e^z\cdot(1-P(z)) = 2z + w \Rightarrow P(z) = 1$, $e^z = 2z + w$ not possible

$\Rightarrow f(z)$ takes the value w infinitely many times.

8(10pts) Let $D = \{|z| < 1, \mathrm{Im}(z) > 0\}$ be the upper half unit-disc. A function f is continuous on the closure $\overline{D}$ and is holomorphic on D. Assume that $f|_{[-1,1]}$ is identically zero where $[-1, 1]$ is the closed interval on the real axis $\mathbb{R} = \{\mathrm{Im}(z) = 0\}$. Prove that f is identically equal to 0.

Use Schwarz reflection principle to extend f to a holomorphic function on $\mathbb{D} = \{|z| < 1\}$:

$$g(z) = \begin{cases} f(z), & \mathrm{Im}\, z \geq 0 \\ \overline{f(\bar{z})}, & \mathrm{Im}\, z < 0 \end{cases}$$

Then $g(z) = 0$ for $z \in [-1, 1]$.

$\Rightarrow$ zeros of $g(z)$ are not isolated in the interior of $\mathbb{D}$

$\Rightarrow$ $g(z) \equiv 0$ on $\mathbb{D}$ $\Rightarrow$ $f(z) \equiv 0$ on $\overline{D}$.

9(10pts)(i) Let $f(z)$ be an entire function. For any $z_0 \in \mathbb{C}$ and $R > 0$, prove the following two mean value formulae:

(1) $$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})d\theta,$$

(2) $$f(z_0) = \frac{1}{\pi R^2}\iint_{D_R(z_0)} f(x+iy)dxdy,$$

where $D_R(z_0) = \{z \in \mathbb{C}; |z - z_0| < R\}$ is the disk of radius R centered at z_0. (Hint: use Cauchy integral formula and then polar coordinate)

(ii) Prove that if an entire function f is integrable (i.e. if $\int_{\mathbb{C}} |f(x+iy)|dxdy < +\infty$), f must be identically equal to zero.

(i) Cauchy Integral Formula: $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta$

Choose $C = \partial D_R = \{ z_0 + Re^{i\theta};\ 0 \le \theta < 2\pi \}$

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{Re^{i\theta}} Re^{i\theta} i\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\, d\theta.$$

$$\Rightarrow \int_0^R f(z_0) r dr = \int_0^R \frac{1}{2\pi}\left(\int_0^{2\pi} f(z_0+re^{i\theta})d\theta\right) r dr = \frac{1}{2\pi}\iint_{D_R} f(x+iy)\, dx dy$$

$$\| \quad f(z_0)\cdot \frac{r^2}{2}\Big|_0^R = f(z_0)\frac{R^2}{2}$$

$$\Rightarrow f(z_0) = \frac{1}{\pi R^2}\iint_{D_R} f(x+iy) dx dy$$

(ii) $$|f(z_0)| \le \frac{1}{\pi R^2}\iint_{D_R} |f(x+iy)|dxdy \le \frac{1}{\pi R^2}\left(\iint_{\mathbb{C}} |f(x+iy)|\, dxdy\right) < +\infty$$

Let $R \to +\infty$, we get $|f(z_0)| \le 0 \Rightarrow |f(z_0)| = 0$.

Since z_0 is arbitrary, we get $f(z) \equiv 0$ on $\mathbb{C}$.