Hadamard Factorization Thm: $\qquad e^{e^z}$ not finite order growth.

f entire. $\forall \varepsilon>0$, $|f(z)| \leq A \cdot B e^{|z|^{\rho+\varepsilon}}$, $\forall z \in \mathbb{C}$

$$f(z) = e^{g} \left(z^m \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k} \right), \quad k=[\rho]$$

zeros at $\{0, a_1, a_2, \cdots\}$

g is a polynomial of degree $\leq k$.

Ex: $\boxed{\dfrac{\sin(\pi z)}{\pi} = z \cdot \prod_{n=1}^{\infty}\left(1-\dfrac{z^2}{n^2}\right)}$ $\qquad \sin(\pi z) = \dfrac{e^{i\pi z}-e^{-i\pi z}}{2i}$, $\rho = 1$.

$$\Downarrow$$

$$\Longleftrightarrow \quad \pi \cdot \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \quad \Longleftarrow \quad \sin(\pi z) = e^{Az+B} \cdot \prod_{n\in\mathbb{Z}} \left(1-\frac{z}{n}\right) \cdot e^{\frac{z}{n}}$$

$\pi \cdot \dfrac{\cos(\pi z)}{\sin(\pi z)}$

$$\pi \cot \pi z = \frac{1}{z}\left(1-\sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} \pi^{2n} \cdot z^{2n}\right)$$

Taylor series at $z=0$

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \frac{1}{z} \cdot \left(1+\underbrace{\sum_{n=1}^{\infty} \frac{2z^2}{z^2-n^2}}\right)$$

$$\sum_{n=1}^{\infty} \frac{2z^2}{z^2-n^2} = -\sum_{n=1}^{\infty} \frac{2z^2}{n^2\left(1-\frac{z^2}{n^2}\right)} = -\sum_{n=1}^{\infty} \frac{2z^2}{n^2} \cdot \sum_{k=0}^{\infty} \frac{z^{2k}}{n^{2k}} = -2 \sum_{k=0}^{+\infty} \sum_{n=1}^{\infty} \frac{z^{2k+2}}{n^{2k+2}} = -2 \sum_{k=0}^{+\infty} \zeta(2k+2) z^{2k+2}$$

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{\pi^{2k} B_k}{(2k)!} \Longleftarrow$$

Ex: e^z-1 zeros at $z = 2\pi i \cdot n$, $n \in \mathbb{Z}$. $|e^z-1| \leq C \cdot e^{|z|}$, $\rho = 1$

$$e^z - 1 = e^{Az+B} \prod_{n\in\mathbb{Z}} \left(1-\frac{z}{2\pi i n}\right) e^{\frac{z}{2\pi i n}} = e^{Az+B} \cdot z \prod_{n=1}^{+\infty} \left(1-\frac{z}{2\pi i n}\right) \cdot \left(1+\frac{z}{2\pi i n}\right)$$

$$= e^{Az+B} \cdot z \prod_{n=1}^{+\infty} \left(1+\frac{z^2}{4\pi^2 n^2}\right)$$

$$1+\frac{z}{2}+\frac{z^2}{6}+\cdots$$

$$\frac{z+\frac{z^2}{2}+\frac{z^3}{6}}{z} = \frac{e^z-1}{z} = e^{Az+B}\cdot\prod_{n=1}^{+\infty}\left(1+\frac{z^2}{4\pi^2n^2}\right)$$

$$\xRightarrow{z=0} 1 = e^B$$

$$1+\frac{z}{2}+\frac{z^2}{6}+\cdots = e^{Az}\cdot\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2n^2}\right) = 1+Az+O(|z|^2). \Rightarrow A=\frac{1}{2}$$

$$e^{Az} = \left(1+Az+\frac{A^2z^2}{2!}+\cdots\right)$$

$$\Rightarrow e^z-1 = e^{\frac{1}{2}z}\, z \prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2n^2}\right).$$

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

$$e^z-1 = \underbrace{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)}_{\sim 2i}\cdot e^{\frac{z}{2}}\cdot(-2i) = (-2i)\cdot e^{\frac{z}{2}}\sin\left(i\frac{z}{2}\right)$$

$$\frac{z}{2} = -i\cdot i\frac{z}{2}, \quad e^{i\cdot i\frac{z}{2}}$$

$$e^{\frac{z}{2}}\cdot z\cdot\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4n^2\pi^2}\right) = e^{\frac{z}{2}}(-2i)\, i\frac{z}{2\pi}\,\pi\cdot\prod_{n=1}^{\infty}\left(1-\frac{\left(i\frac{z}{2\pi}\right)^2}{n^2}\right).$$

- <u>Show $e^z-z=0$ has infinitely many solutions in $\mathbb{C}$.</u>

$$e^z-z = e^{Az+B}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}$$

If only finitely many zeros, then $e^z-z = e^{Az}\cdot P(z)\quad\forall z\in\mathbb{C}$

$$e^z-1 = e^{Az}\cdot A\cdot P+e^{Az}\cdot P' = e^{Az}\cdot Q$$

$$e^z = e^{Az}\cdot R \Rightarrow A=1,\ R=1.$$

$$e^z-z = e^z\cdot P \Rightarrow (1-P)e^z = z\quad\forall z$$

$$\Rightarrow 1-P=0 \Rightarrow z=0 \rightarrow\leftarrow$$

- If Growth order of $\overset{\text{entire}}{f}$ is not an integer, then f takes any value infinitely many times.

$$f(z)-w = e^{g(z)} \cdot p(z) \quad \text{growth order is an integer} = \deg g$$

an integer.

- Every meromorphic on $\mathbb{C}$ is a quotient of two entire fcts.

Can prescribe $\{0\} \cup \{a_n\}$ as the zero set of an entire fct $\Leftarrow$ Weierstrass product.

Every meromorphic fct. on $\mathbb{CP}^1$ is a rational fct. $= \dfrac{P}{Q}$.

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = S^2$$

- Gamma function $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$.

extends analytically to a meromorphic fct. on $\mathbb{C}$ with poles at $n = 0, -1, -2, \cdots$ with residue $(-1)^n \cdot \frac{1}{n!}$

$\Gamma(s+1) = s \cdot \Gamma(s)$.

$$\boxed{\Gamma(s) \cdot \Gamma(1-s) = \frac{\pi}{\sin \pi s}} \Leftarrow \overset{\boxed{0<s<1}}{\Gamma(1-s)\cdot\Gamma(s)} = \boxed{\int_0^\infty \frac{x^{s-1}}{1+x}\, dx} = \frac{\pi}{\sin \pi s}.$$

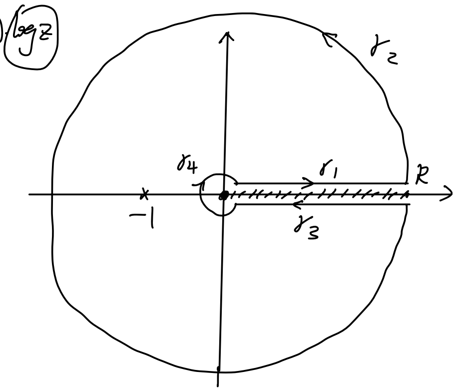
poles $0, -1, -2, \cdots$ $1, 2, 3, \cdots$

$\| \; x = e^t$

$$\int_{-\infty}^{+\infty} \frac{e^{(s-1)t} e^t dt}{1+e^t} = \boxed{\int_{-\infty}^{+\infty} \frac{e^{st}\, dt}{1+e^t}}$$

Ex. 2 of 2.1 III

$$I = \int_0^\infty \frac{x^{s-1}}{1+x}\,dx . \qquad f(z) = \frac{z^{s-1}}{1+z}, \quad z^{s-1} = e^{(s-1)\log z}$$



$$\mathrm{res}_{z=-1} \frac{z^{s-1}}{\underset{z-(-1)}{1+z}} = (-1)^{s-1} = e^{(s-1)(\pi i)} = -e^{s\pi i}$$

$$\int_{\gamma_1} f(z)\,dz = \int_\varepsilon^R \frac{x^{s-1}}{1+x}\,dx \xrightarrow[R\to+\infty]{\varepsilon\to0} I$$

$$\int_{\gamma_3} f(z)\,dz = \int_R^\varepsilon \frac{e^{(s-1)(\log x+2\pi i)}}{1+x}\,dx = -\int_\varepsilon^R \frac{x^{s-1}}{1+x}\,dx \cdot \underbrace{e^{2\pi(s-1)\cdot i}}_{=e^{2\pi i s}\cdot e^{-2\pi i},\ e^{-2\pi i}=1}$$

$$\downarrow$$

$$-I\cdot e^{i\cdot 2\pi s}$$

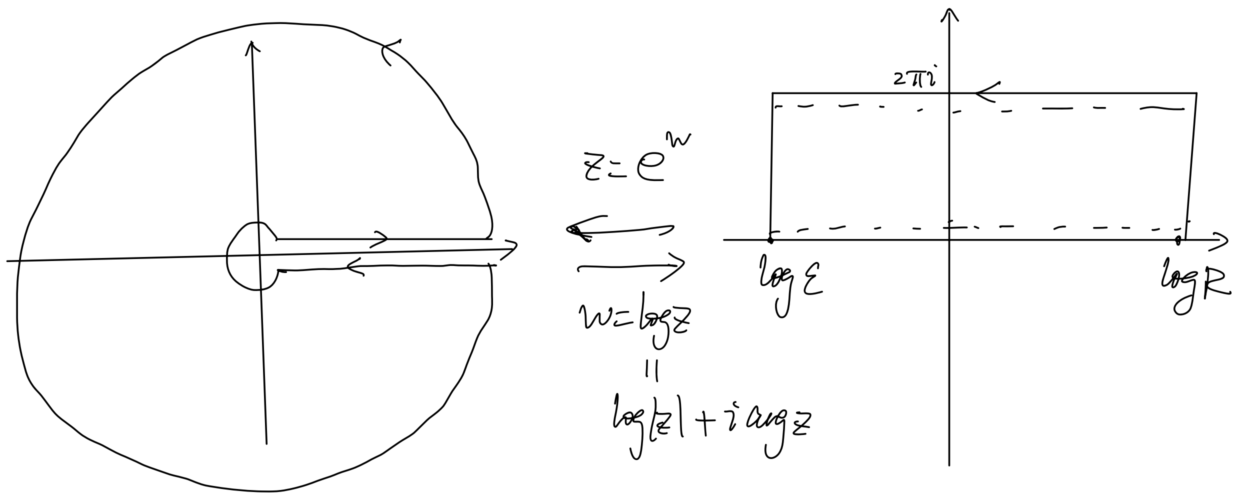
$$I - I\cdot e^{i\cdot 2\pi s} = 2\pi i\cdot \mathrm{res}_{z=-1} f(z) = 2\pi i\left(-e^{s\pi i}\right)$$

$$\Rightarrow I = -\frac{2\pi i\cdot e^{s\pi i}}{1-e^{i\cdot 2\pi s}} = \underbrace{\left(\frac{2i}{e^{s\pi i}-e^{-s\pi i}}\right)}_{=\frac{1}{\sin(\pi s)}}\cdot\pi = \frac{\pi}{\sin\pi s}$$

$$|z|^{s-1} = \left|e^{(s-1)\log z}\right|$$

$$\left|\int_{\gamma_2} \frac{z^{s-1}}{1+z}\,dz\right| \le \int_{\gamma_2} \frac{|z^{s-1}|}{|z|-1}\,|dz| \le R^{s-1}\cdot\frac{1}{R-1}\,2\pi R \xrightarrow[R\to+\infty]{0\le s<1} 0$$

$$\left|\int_{\gamma_4} \frac{z^{s-1}}{1+z}\,dz\right| \le \int_{\gamma_4} \frac{|z|^{s-1}}{1-|z|}\,|dz| = \frac{\varepsilon^{s-1}}{1-\varepsilon}\cdot 2\pi\varepsilon = \frac{2\pi\varepsilon^s}{1-\varepsilon} \xrightarrow[s>0]{\varepsilon\to0} 0$$



$\frac{1}{\Gamma(s)}$ zeros at $0, -1, -2, \ldots$

$= \Gamma(1-s) \cdot \frac{\sin \pi s}{\pi}$

$$\frac{1}{\Gamma(s)} = e^{As+B} s \cdot \prod_{n=1}^{\infty} \left(1 - \frac{s}{-n}\right) e^{-\frac{s}{n}}$$

$$= e^{As+B} \cdot s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

$$\frac{1}{\Gamma(s) \cdot s} = e^{As+B} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \cdot e^{-\frac{s}{n}}$$

$\downarrow s \to 0$ $\qquad$ $\downarrow s \to 0$

$\frac{1}{1} = 1$ $\qquad$ e^B

$$1 = e^A \cdot \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \cdot e^{-\frac{1}{n}} \Rightarrow e^A = e^{+\gamma}$$

$$= e^{\sum_{n=1}^{N} \left(\ln\left(1 + \frac{1}{n}\right) - \frac{1}{n}\right)} = \ln N - \sum_{n=1}^{N} \frac{1}{n} \xrightarrow{N \to +\infty} -\gamma$$

$$\Rightarrow \frac{1}{\Gamma(s)} = e^{\gamma \cdot s} s \cdot \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-\frac{s}{n}}$$

$$-\log \Gamma(s) = \gamma \cdot s + \log s + \sum_{n=1}^{\infty} \left(\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right)$$

$$\frac{d}{ds} \log \Gamma = -\frac{\Gamma'}{\Gamma} = \gamma + \frac{1}{s} + \sum_{n=1}^{\infty} \Big(\underbrace{\frac{\frac{1}{n}}{1+\frac{s}{n}} - \frac{1}{n}}_{\frac{1}{n+s} - \frac{1}{n} = \frac{-s}{(n+s)\cdot n}}\Big) = \gamma + \sum_{n=1}^{\infty} \frac{-s}{n(n+s)}$$

$$\Rightarrow \frac{d}{ds} \log \Gamma = -\gamma + \sum_{n=1}^{\infty} \frac{s}{(n+s)n}, \quad \frac{d}{ds}\left(\frac{\Gamma'}{\Gamma}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+s)^2}$$

$$\cdot \quad \frac{d}{dx} \int_x^{x+1} \log \Gamma(t)\, dt = \log \overset{x\cdot\Gamma(x)}{\overset{\parallel}{\Gamma(x+1)}} - \log \Gamma(x) = \log x \quad \forall x > 0.$$

$$\Rightarrow \int_x^{x+1} \log \Gamma(t)\, dt = x \cdot \log x - x + C. \quad x > 0$$

$$\underset{\underset{\log \Gamma(n)}{\rotatebox{90}{$\leq$}}}{\int_n^{n+1} \log \Gamma(t)\, dt} = n \cdot \log n - n + C \leq \underbrace{\log \Gamma(n+1)}_{\log \Gamma(n) + \log n.}$$

$$\Rightarrow \quad \log \Gamma(n) \sim n \cdot \log n + O(n) \quad \text{weak Stirling}$$

$$\log n + \log \Gamma(n) = \log \Gamma(n+1) \sim n \cdot \log n + O(n)$$

Strong Stirling:

$$\log \Gamma \sim \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi$$

$$\Gamma(n) \sim n^{n-\frac{1}{2}} \cdot e^{-n} \cdot (2\pi)^{\frac{1}{2}}$$

$$\begin{array}{c} \Gamma(n) \\ \parallel \\ \frac{1}{n}\cdot n! \end{array}$$

$$\Rightarrow n! \sim n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ has pole at $\underline{s=1}$.

trivial zeros at $\underline{-2, -4, -6, \dots, -2k, \dots)}$

$$\xi(s) = s\underline{(s-1)} \cdot \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

zeros of ξ = nontrivial zeros of $\zeta = \{ \rho$ zeros of ζ, $0 < \mathrm{Re}\,\rho < 1 \}$

$$\Rightarrow \quad \xi(s) = \xi(1-s)$$

has a order of growth 1.

$$\Rightarrow \quad \xi(s) = e^{As+B} \prod_{\rho \in \text{Zeros of } \xi} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

$$\zeta(s) = \prod_{p \text{ primes}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{Euler product formula.}$$

$$\log \zeta(s) = -\sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^s}\right) = -\sum_{p \text{ prime}} \sum_{k \geq 0} \frac{1}{p^{ks}}$$

$$\boxed{\frac{\zeta'}{\zeta} = -\sum_{p \text{ prime}} \sum_{k \geq 0} \frac{\log p^k}{p^{ks}}}$$

$$\boxed{\frac{\xi'}{\xi} = A + \sum_{\rho \text{ zeros}} \left(\frac{-\frac{1}{\rho}}{1-\frac{z}{\rho}} + \frac{1}{\rho} \right)}$$

$$= A + \sum_{\rho} \left(\frac{1}{z-\rho} + \frac{1}{\rho} \right)$$