Schwarz Lemma: $f: \mathbb{D} \to \mathbb{D}$, holomorphic. $f(0)=0$.

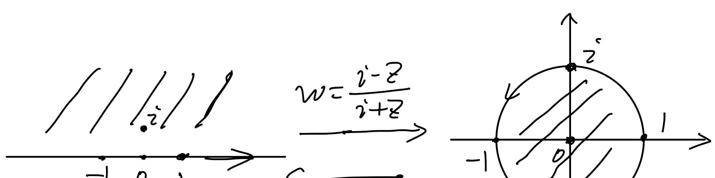
$$|f(z)| \le |z|, \quad |f'(0)| \le 1.$$

If either $|f(z_0)| = |z_0|$ $z_0 \ne 0$, or $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ is a rotation.

$\Longrightarrow$ $\mathrm{Aut}(\mathbb{D}) = \left\{ \psi_\alpha = \dfrac{\alpha - z}{1 - \bar{\alpha} z}, \ |\alpha| < 1 \right\}$, $\quad \psi_\alpha(0) = \alpha$
$\alpha \in \mathbb{D}$ $\quad \psi_\alpha(\alpha) = 0$

$\simeq SU(1,1)$ $\quad \psi_\alpha \circ \psi_\alpha = Id_{\mathbb{D}}$

. $\mathbb{H} \longrightarrow \mathbb{D}$



$z = i \cdot \dfrac{1-w}{1+w}$

$iw + wz = i - z$

$wz + z = i - iw$
$\quad \| $
$w+1$

$PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \{\pm I_2\}$
$\|$

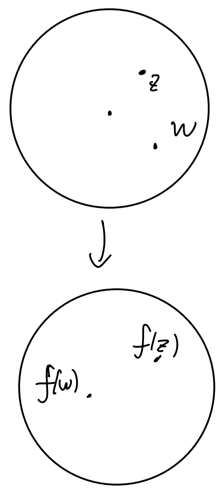
$\mathrm{Aut}(\mathbb{H}) = \left\{ \dfrac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ 2x2 real}, \ a,b,c,d \in \mathbb{R}, \ ad-bc>0 \right\}$

- VII 10. $F: \mathbb{H} \to \mathbb{C}$. $|F(z)| \le 1$, $F(i) = 0$.

$$\begin{tikzcd} \mathbb{H} \arrow[r] & \mathbb{C} \\ \mathbb{D} \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\Big|F\Big(\underbrace{i\tfrac{1-w}{1+w}}_{= z}\Big)\Big| \le |w| \Rightarrow |F(z)| \le \left|\frac{i-z}{i+z}\right|. \ \forall z \in \mathbb{H}.$$

13.



$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right| = |\varphi_w(z)|$$

$$\varphi_w(z) = \frac{w-z}{1-\bar{w}z}.$$

$$\boxed{\rho(f(z),f(w)) \le \rho(z,w).}$$

$$\rho(f(z),f(w)) = |\varphi_{f(w)}(f(z))|, \qquad \rho(z,w) = |\varphi_w(z)|, \qquad \varphi_w(z) = u$$

$$\left|\underbrace{(\varphi_{f(w)}\circ f\circ \varphi_w^{-1})}(u)\right| \le |u| \qquad ✓$$

$$0 \xrightarrow{\varphi_w^{-1}} w \xrightarrow{f} f(w) \xrightarrow{\varphi_{f(w)}} 0$$

$$\left(d_{hyp}(z,w) = d_{hyp}(\varphi_w(z), \varphi_w(w)) = d_{hyp}(\varphi_w(z), 0) = \big|\ln|\varphi_w(z)|\big| . \right)$$

(b) Schwarz-Pick lemma: $\dfrac{f'(z)}{1-|f(z)|^2} \le \dfrac{1}{1-|z|^2}$

$$g(z) = (\varphi_{f(w)}\circ f\circ \varphi_w^{-1})(0) = 0. \qquad \mathbb{D}\to\mathbb{D}$$

$$\varphi_w^{-1} = \varphi_w$$

$$\Rightarrow \boxed{|g'(0)| \le 1}. \qquad g' = \varphi_{f(w)}'(f(w))\cdot f'(w)\cdot \varphi_w'(0).$$

$$\varphi_w(z) = \frac{w-z}{1-\bar{w}z}, \quad \varphi_w' = \frac{(-1)\cdot(1-\bar{w}z) - (w-z)\cdot(-\bar{w})}{(1-\bar{w}z)^2}$$

$$\varphi_w'(0) = \frac{-1+|w|^2}{1}, \quad \varphi_w'(w) = \frac{-(1-|w|^2)+0}{(1-|w|^2)^2} = -\frac{1}{1-|w|^2}$$

$$\left| -\frac{1}{1-|f(w)|^2}\cdot f'(w)\cdot(-1+|w|^2)\right| \le 1 \Rightarrow \boxed{\frac{f'(w)}{1-|f(w)|^2} \le \frac{1}{1-|w|^2}, \ \forall w\in\mathbb{D}}$$

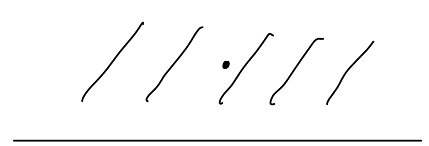
$$g_{hyp} = \frac{|dz|^2}{(1-|z|^2)^2} \qquad \text{hyperbolic metric}$$

$$f^* g_{hyp} = \frac{|df|^2}{(1-|f(z)|^2)^2} = \frac{|f'|^2 |dz|^2}{(1-|f(z)|^2)^2} \overset{\text{Schwarz-Pick}}{\leq} \frac{|dz|^2}{(1-|z|^2)^2} = g_{hyp}.$$

$$g(z) = \psi_{f(w)} \circ f \circ \psi_w^{-1} \qquad |g'(0)| = 1$$

$$\Rightarrow \quad \psi_{f(w)} \circ f \circ \psi_w^{-1}(z) = e^{i\theta}(z).$$

$$\Rightarrow \quad f = \psi_{f(w)}^{-1} \circ e^{i\theta} \circ \psi_w \text{ is an automorphism of } \mathbb{D}.$$

15: 

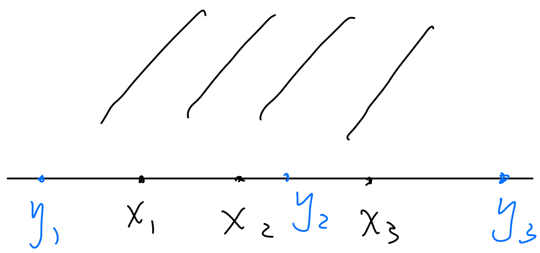
$$\mathrm{Aut}(\mathbb{H}) = \left\{ T_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a,b,c,d \in \mathbb{R}, \ ad-bc>0 \right\} = PSL(2,\mathbb{R})$$

$$\#\{T_\gamma(z) = z\}, \quad T_\gamma \neq Id, \quad z \in \mathbb{C}$$

- $z, \bar{z}$, $z \in \mathbb{H}$ — hyperbolic rotation
- $x_1, x_2 \in \mathbb{RP}^1$. — hyperbolic translation
- $x \in \mathbb{RP}^1$ — parabolic translation (limit rotation)

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0$$

$$(d-a)^2 + 4bc = \begin{cases} >0 & \text{hyp. translation} \\ =0 & \text{parabolic translation} \\ <0 & \text{hyp. rotation.} \end{cases}$$



$\exists$ unique $T_\gamma \in \mathrm{Aut}(\mathbb{H})$

$$T_\gamma(x_i) = y_i$$

$$\overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$\frac{a x_i + b}{c x_i + d} = y_i, \quad i = 1, 2, 3.$$

$\boxed{\text{Cross ratio}}$:

$$(z_1, z_2; z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}$$

$$\shortparallel$$

$$(T_\gamma(z_1), T_\gamma(z_2); T_\gamma(z_3), T_\gamma(z_4))$$

$$(z, x_1; x_2, x_3) = (\underset{\shortparallel}{\overset{w}{T_\gamma(z)}}, y_1; y_2, y_3) \Longrightarrow w = T_\gamma(z).$$

$$\frac{z - x_2}{z - x_3} \Big/ \frac{x_1 - x_2}{x_1 - x_3} \qquad \frac{w - y_2}{w - y_3} \Big/ \frac{y_1 - y_2}{y_1 - y_3}$$

- Fact: linear fractional transformation maps circles to circles

$$\mathrm{Aut}(\mathbb{C} \cup \{\infty\}) \cong \left\{ \boxed{\frac{az+b}{cz+d}} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \right\} \quad (\text{lines} = \text{circles passing through } \infty)$$

$$\overset{\shortparallel}{\mathrm{Aut}(S^2)} \cong PGL(2, \mathbb{C})$$

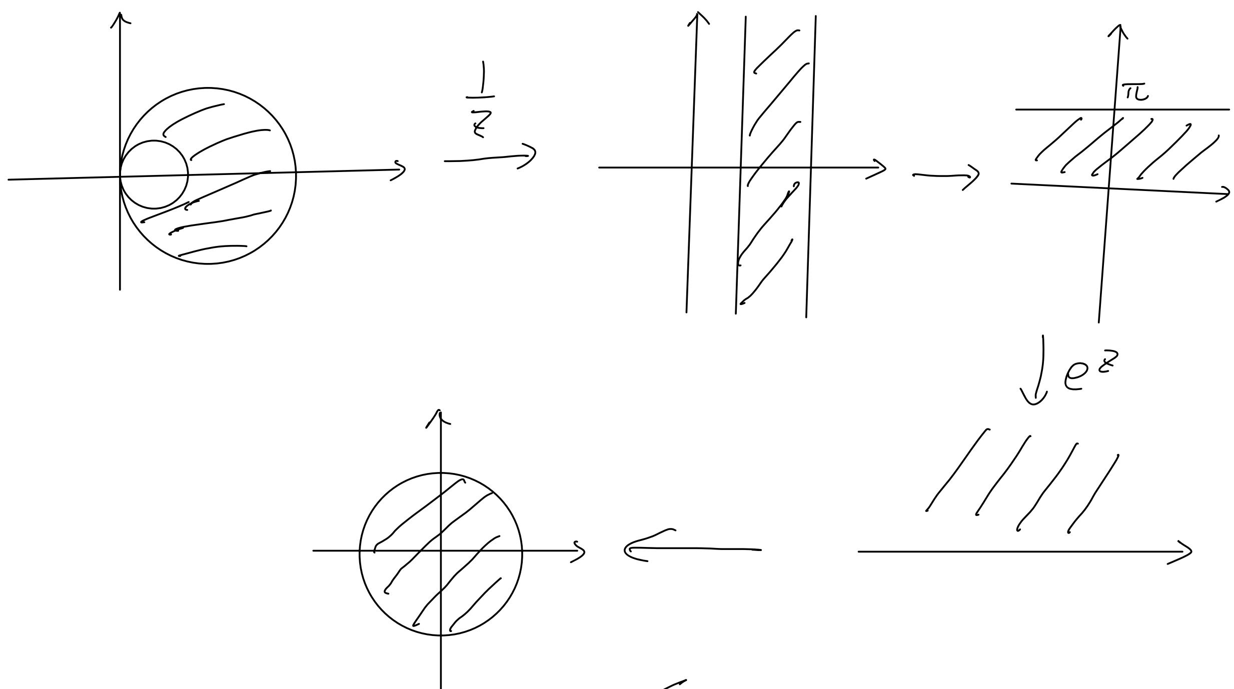
Fact: Any meromorphic function on $\overset{S^2}{\overset{\shortparallel}{\underline{\mathbb{C} \cup \{\infty\}}}}$ is rational

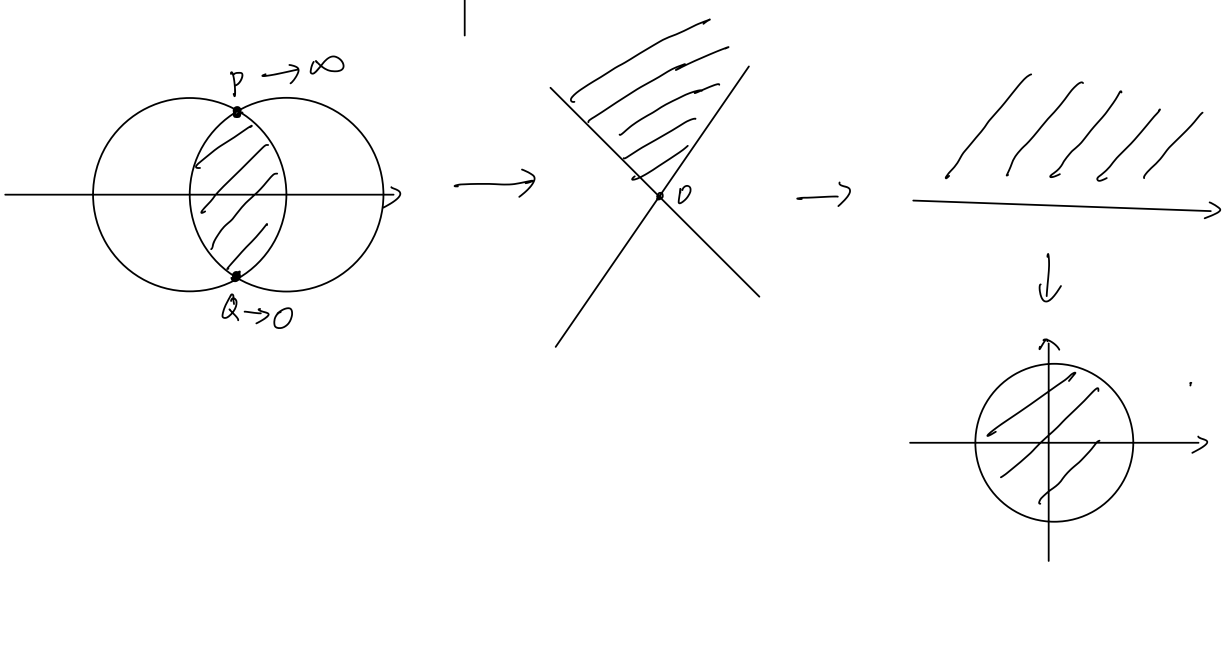
$$\frac{P(z)}{Q(z)}.$$

$f(z)$ has a pole at $z_0 \in \mathbb{C} \iff |f(z)| \to +\infty$ as $z \to z_0$

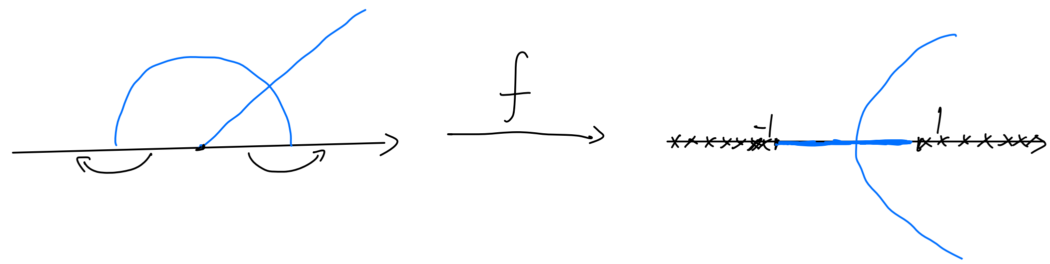
$\iff f(z) = \dfrac{g(z)}{(z-z_0)^k}$ g holomorphic near z_0.

pole at $\infty \in \mathbb{C} \iff f\left(\frac{1}{z}\right)$ has a pole at 0.



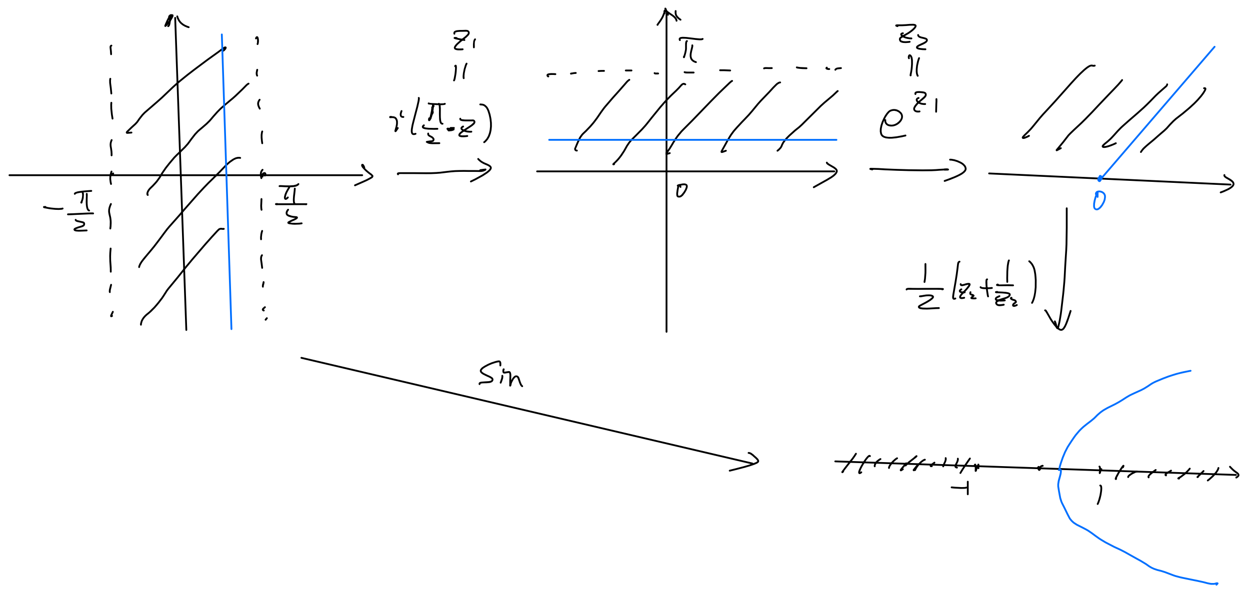


Ex: $$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\,\frac{z^2+1}{z}$$

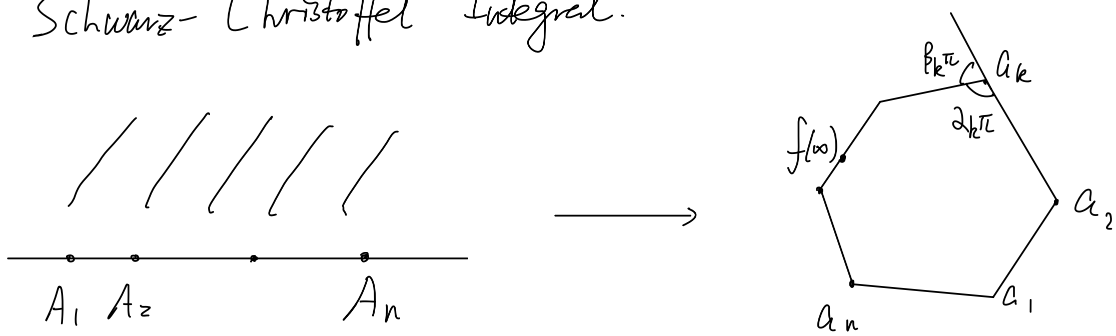


$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \cos\left(\frac{\pi}{2} - z\right) = \frac{e^{i(\frac{\pi}{2}-z)} + e^{-i(\frac{\pi}{2}-z)}}{2} = \frac{1}{2}\cdot\left(e^{i(\frac{\pi}{2}-z)} + \frac{1}{e^{i(\frac{\pi}{2}-z)}}\right)$$



Schwarz-Christoffel Integral.



$$f(z) = C_1 \int_0^z \frac{dz}{(z-A_1)^{\beta_1} \cdots (z-A_n)^{\beta_n}} + C_2$$

$\alpha_k + \beta_k = 1.$

$\sum_{k=1}^{n} \beta_k = 2.$

$0 < \beta_k < 1.$

$\underline{\text{VIII}}$: 20.

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}$$

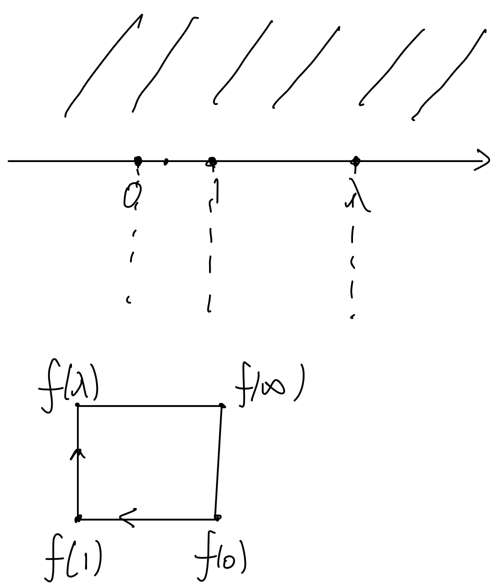
$\lambda > 1$

$$\frac{d\zeta}{\zeta^{\frac{1}{2}} \underline{(\zeta-1)^{\frac{1}{2}}}\ \underline{(\zeta-\lambda)^{\frac{1}{2}}}}$$

$\beta_k = \frac{1}{2}, \ \alpha_k = \frac{1}{2}$

$\zeta = x$

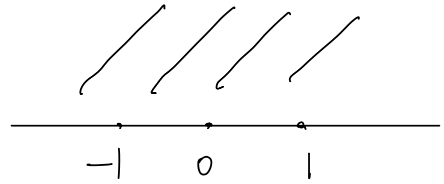
$$(\zeta - A_k)^{\beta_k} = \begin{cases} (x-A_k)^{\beta_k} & x > A_k \\ (A_k - x)^{\beta_k} e^{i\beta_k\pi} & x < A_k \end{cases}$$

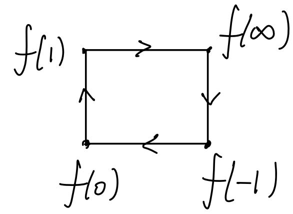


$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

$$\parallel$$

$$\frac{d\zeta}{(\zeta+1)^{\frac12}\,\zeta^{\frac12}\,(\zeta-1)^{\frac12}}$$





$$\int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} = \int_0^1 x^{-\frac12}(1-x^2)^{-\frac12}\,dx$$

$$\underset{x=t^{\frac12}}{\overset{t=x^2}{=}} \int_0^1 t^{-\frac14}(1-t)^{-\frac12}\,\frac12 t^{-\frac12}\,dt = \frac12 \int_0^1 t^{-\frac34}(1-t)^{-\frac12}\,dt$$

$$dx = \frac12 t^{-\frac12}\,dt$$

$$= \frac12 B\left(\frac14, \frac12\right) = \frac12 \frac{\Gamma\left(\frac14\right)\Gamma\left(\frac12\right)}{\Gamma\left(\frac34\right)} = \frac12 \frac{\Gamma\left(\frac14\right)\cdot\sqrt{\pi}}{\frac{\sqrt2\,\pi}{\Gamma\left(\frac14\right)}} = \frac{\Gamma\left(\frac14\right)^2}{2\sqrt{2\pi}}$$

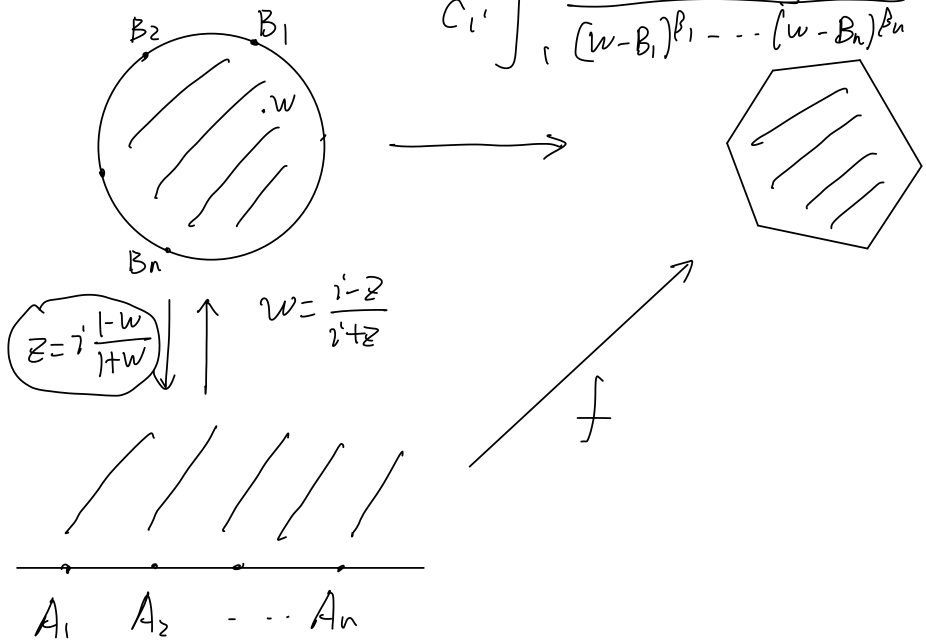
$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}\cdot(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\boxed{\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}}$$

$$\Gamma\left(\frac12\right)^2 = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi \Rightarrow \Gamma\left(\frac12\right) = \pi$$

$$\Gamma\left(\frac14\right)\cdot\Gamma\left(\frac34\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi}{\frac{1}{\sqrt2}} = \sqrt2\,\pi$$

$$C_1' \int_1^w \frac{d\zeta}{(w-B_1)^{\beta_1}\cdots(w-B_n)^{\beta_n}} + C_2.$$



$$f(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta-A_1)^{\beta_1}\cdots(\zeta-A_n)^{\beta_n}} + C_2 \qquad \sum \beta_k = 2$$

$$\parallel$$

$$C_1' \int_0^z \frac{-\frac{2\,dw}{\cancel{(1+w)^2}}}{\prod_{k=1}^{n} \frac{(w-B_k)^{\beta_k}}{\cancel{(1+w)^{\beta_k}}}} + C_2'$$

$$\parallel$$

$$C_1' \int_1^w \frac{dw}{(w-B_1)^{\beta_1}\cdots(w-B_n)^{\beta_n}} + C_2'$$

$$\zeta = i\frac{1-w}{1+w} = i\frac{2-(1+w)}{1+w}$$

$$A_k = i\frac{1-B_k}{1+B_k}$$

$$d\zeta = -\frac{2}{(1+w)^2}dw$$

$$\zeta - A_k = i\left(\frac{1-w}{1+w} - \frac{1-B_k}{1+B_k}\right) = i\frac{-2(w-B_k)}{(1+w)(1+B_k)}$$

$$(1-w)(1+B_k) - (1-B_k)(1+w)$$

$$\parallel$$

$$(\cancel{1} + B_k - w - \cancel{B_k w}) - (\cancel{1} + w - B_k - \cancel{B_k w})$$

$$\parallel$$

$$-2(w-B_k)$$

Weierstrass Product $\qquad$ Hadamard Factorization

$f: \mathbb{C} \to \mathbb{C}$ entire function.

$\{0\} \cup \{a_1, a_2, \dots\}$ zeros of f.

$|a_k| \to +\infty$

$$z^m \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \cdot e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}}$$

$$z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right) + \cdots + \frac{1}{k}\left(\frac{z}{a_k}\right)^{k}} \text{ converges.}$$

$$\|$$

$$z^m \cdot e^{\sum_{k=1}^{\infty} \left(\log\left(1 - \frac{z}{a_k}\right) + \frac{z}{a_k} + \cdots + \frac{1}{k}\left(\frac{z}{a_k}\right)^k \right)}$$

$$\|$$

$$\left| -\frac{1}{k+1}\left(\frac{z}{a_k}\right)^{k+1} - \frac{1}{k+2}\left(\frac{z}{a_k}\right)^{k+2} - \cdots \right|$$

$$\wedge\!\backslash$$

$$\sum_k \frac{1}{k+1} \frac{|z|^{k+1}}{|a_k|^{k+1}} \quad \text{converges}$$

$|a_k| \to +\infty$

$$\sum_{k=1}^{\infty} \frac{1}{m_k + 1} \left(\frac{z}{a_k}\right)^{m_k + 1}$$

If $\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\rho}}$ converges, then $z^m \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \cdots + \frac{1}{[\rho]}\left(\frac{z}{a_k}\right)^{[\rho]}}$ converges to an entire fct.