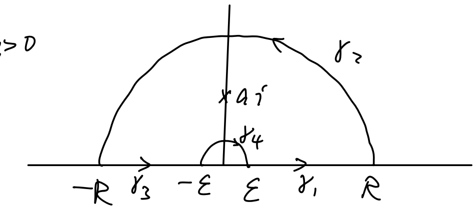
III 10. $\displaystyle\int_0^\infty \frac{\log x}{x^2+a^2}\,dx$, $a>0$



$$\int_{\Gamma_{\varepsilon,R}} \underbrace{\frac{\log z}{z^2+a^2}}_{= f(z)}\,dz = 2\pi i\cdot \mathrm{res}_{z=ai}\left(\frac{\log z}{z^2+a^2}\right)$$

$$\mathrm{res}_{ai}\frac{\log z}{(z+ai)(z-ai)} = \frac{\log(ai)}{(ai+ai)} = \frac{\log a + i\cdot\frac{\pi}{2}}{2ai}$$

$$\left(\frac{1}{(n-1)!}\frac{d^n}{dz^n}(z-z_0)^{n-1}f(z)\Big|_{z=z_0}\right)$$

$$a_{-n}+a_{-(n-1)}(z-z_0)+\cdots+a_{-1}(z-z_0)^{n-1}$$

$$= \cancel{2}\pi \cancel{i}\cdot\frac{\left(\log a + i\cdot\frac{\pi}{2}\right)}{\cancel{2}a\cancel{i}} = \frac{\pi}{a}\left(\log a + i\cdot\frac{\pi}{2}\right) = \frac{\pi\cdot\log a}{a} + \frac{\pi^2}{2a}i$$

$$\int_{\gamma_1} f(z)\,dz = \int_\varepsilon^R \frac{\log x}{x^2+a^2}\,dx \xrightarrow[R\to+\infty]{\varepsilon\to 0} \int_0^\infty \frac{\log x}{x^2+a^2}\,dx$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \left|\int_0^\pi \frac{\log(Re^{i\theta})}{R^2e^{2i\theta}+a^2}\,Re^{i\theta}\cdot i\,d\theta\right| \xrightarrow{R\to+\infty} 0$$

$\log(Re^{i\theta}) = \log R + i\theta$, $\quad R^2e^{2i\theta}+a^2 \sim R^2$

$\gamma_2 = \{Re^{i\theta},\ 0\le\theta\le\pi\}$

$$\le C\cdot\frac{R\cdot\log R}{R^2} \xrightarrow{R\to+\infty} 0$$

$$\int_{\gamma_3} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{\log x}{x^2+a^2}\,dx = \int_R^\varepsilon \frac{(\log x + i\pi)}{x^2+a^2}(-dx)$$

$\log x = \log|x| + i\cdot\pi$, $\quad |x| = -x$

$$= \int_\varepsilon^R \frac{\log x}{x^2+a^2} + i\cdot\int_\varepsilon^R \frac{\pi}{x^2+a^2}\,dx$$

$$\xrightarrow[R\to+\infty]{\varepsilon\to 0} I + i\int_0^{+\infty}\frac{\pi}{x^2+a^2}\,dx$$

$$= i\cdot\pi\int_0^{+\infty}\frac{1}{\left(\frac{x}{a}\right)^2+1}\,\frac{dx}{a}\cdot\frac{1}{a} = i\,\frac{\pi^2}{2a}$$

$$\int_0^{+\infty}\frac{dt}{t^2+1} = \tan^{-1}(t)\Big|_0^{+\infty} = \frac{\pi}{2}$$

$$\left|\int_{\gamma_4} \frac{\log z}{z^2+a^2}\,dz\right| = \left|\int_{\pi}^{0} \frac{\log \varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+a^2}\,\varepsilon e^{i\theta}\cdot i\cdot d\theta\right|$$

$\gamma_4 = \{\varepsilon\cdot e^{i\theta} : \theta : \pi \to 0\}$

$$\leq \int_0^{\pi} \frac{\sqrt{|\log \varepsilon|^2+\theta^2}}{a^2-\varepsilon^2}\,\varepsilon\,d\theta \xrightarrow{\varepsilon\to 0} 0$$

$$\varepsilon\cdot\log\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

$$\Rightarrow \int_{T_{\varepsilon,R}} f(z)\,dz \xrightarrow[R\to+\infty]{\varepsilon\to 0} 2\cdot\int_0^{+\infty} \frac{\log x}{x^2+a^2}\,dx + i\cdot\frac{\pi^2}{2a}$$

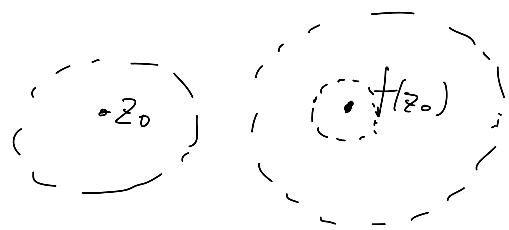
$$\int_{T_{\varepsilon,R}} f(z)\,dz = \frac{\pi\cdot\log a}{a} + i\cdot\frac{\pi^2}{2a}$$

$$\Rightarrow \int_0^{+\infty} \frac{\log x}{x^2+a^2}\,dx = \frac{\pi\cdot\log a}{2a}$$

- Maximum Principle.

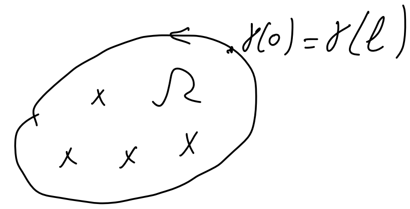
f hol. non-constant on Ω Then $|f(z)|$ does not obtain local maximum in the interior of Ω

- Open mapping Thm: f maps open sets to open sets.



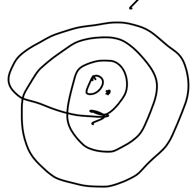
- Rouché's Thm: f, g hol. in open disc D, $\partial D = C$

$|f(z)| > |g(z)| \Rightarrow f$ and $f \pm g$ have the same number of zeros.



- Argument principle: $f \neq 0$ on $\partial\Omega$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz = \#\{\text{zeros}\} - \#\{\text{poles}\}$$



$$d \log f = d(\log|f|) + i \cdot d \arg f$$

$$\log|f| \Big|_{\gamma(0)}^{\gamma(l)} = 0 \qquad \arg f \Big|_{\gamma(0)}^{\gamma(l)}$$

III 15.C $\quad w_1, \dots, w_n \in S^1 = \{|z|=1\}$

Prove $\underline{\exists z \in S^1}$ s.t. $\prod_{i=1}^{n} |z-w_i| \geq 1$.

$$\left(\exists z \in S^1 \text{ s.t. } \prod_{i=1}^{n} |z-w_i| = 1 \right)$$

$$f(z) = \prod_{i=1}^{n} (z-w_i) \qquad |f(0)| = \Big|\prod_i w_i\Big| = 1 \leq \overset{M}{\overset{\shortparallel}{\max_{|z|=1} |f(z)|}}$$

$f(w_i) = 0$. $\qquad |f(z)|\Big|_{S^1} : S^1 \to \mathbb{R}$

16: f, g are hol. in $\Omega \supseteq \{|z| \leq 1\}$

Suppose f has a simple zero at $z=0$ and vanishes nowhere else in $\{|z| \leq 1\}$. $\quad f_\varepsilon(z) = f(z) + \varepsilon g(z)$. $\quad f(z_\varepsilon) = 0$

If ε suff. small, then $f_\varepsilon(z)$ has a unique zero in $|z| \leq 1$.

$$|f(z)| \geq \delta > \varepsilon \cdot |g(z)| \quad \text{on } \{|z|=1\}$$

$\Rightarrow \# \{\text{zeros of } f_\varepsilon \text{ in } |z|<1\} = \# \{\text{zeros } f \text{ in } |z|<1\} = 1.$

17: f non-constant hol. $\overset{\text{open}}{\Omega} \supset \{|z|=1\}$

If $|f(z)|=1$ whenever $|z|=1$, then the image of f contains the unit disc $= \mathbb{D}$

Proof: $w_0 \in \mathbb{D}$, $f(z)=w_0$ must have a root ?

on $\partial\mathbb{D}$ $|f(z)|=1>w_0 \Rightarrow f(z)-w_0$ has the same # of zeros as $f(z)$

Enough to show $f(z)=0$ must have a root. $|f(z)|\le 1$ on $\mathbb{D}$

suppose not. Then $\left|\frac{1}{f(z)}\right| \geq 1$ on $\mathbb{D}$

$\left|\frac{1}{f(z)}\right| = 1$ on $\partial\mathbb{D}$

$\Longrightarrow \frac{1}{f(z)} = \text{constant}$ contradiction.

(b) If $|f(z)| \not\geq 1$ when $|z|=1$. and

$\exists z_0 \in \mathbb{D}$ s.t. $|f(z_0)| < 1$. Then $f(\Omega) \supset \mathbb{D}$

$f(z)=0$ must have a root

$\Downarrow$

$f(z)-w_0$ must have a root $\forall w_0 \in \mathbb{D}$

Suppose not. Then $\left|\frac{1}{f(z)}\right| \leq 1$ when $|z|=1$

$$\frac{1}{|f(z_0)|} > 1$$

$$\Rightarrow \max_{\partial \mathbb{D}} \left|\frac{1}{f(z)}\right| < \max_{\mathbb{D}} \left|\frac{1}{f(z)}\right|$$

$$\Rightarrow \frac{1}{f(z)} = \text{const.} \Rightarrow f(z) = \text{const.}$$ Contradiction

Maximum Principle: $\max_{\partial \Omega} |f(z)| = \max_{\overline{\Omega}} |f(z)|$

If $f(z_0) = \max_{\overline{\Omega}} |f(z)|$ for $z_0 \in \Omega$, then $f = \text{const.}$

Conformal Mapping: $f: \Omega_1 \to \Omega_2$ bijective holomorphic map

$\Downarrow$

$f'(z) \neq 0$ for any $z \in \Omega_1$

f is locally injective at z_0 $\Longleftrightarrow$ $\boxed{f'(z_0) \neq 0}$ $\Longrightarrow$ locally conformal

$\uparrow$

$f: B_\varepsilon(z_0) \to \mathbb{C}$ is injective.

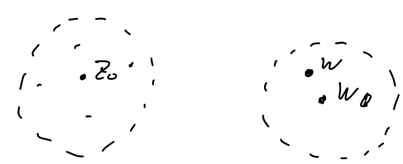
proof: $f'(z_0) \neq 0$. $\quad f(z) = \underset{w_0}{\overset{\|}{f(z_0)}} + \underline{f'(z_0)\cdot(z-z_0)} + \overset{O(|z-z_0|^2)}{\overset{\|}{g(z)}}$

$\#\{\text{zeros of } f(z) - w\} = \#\{\text{zeros of } f(z_0) + f'(z_0)(z-z_0) - w\}$

$O(|z-z_0|^2) = |g(z)| < |\underset{w_0}{\underset{\|}{f(z_0)}} + f'(z_0)(z-z_0) - w|$

$f'(z_0) = 0$, $\quad \mathrm{ord}_{z_0}(f - f(z_0)) = k$. $\quad f(z) = f(z_0) + (z-z_0)^k \frac{f^{(k)}(z_0)}{k!} + \overset{O(|z-z_0|^{k+1})}{\overset{\|}{g(z)}}$

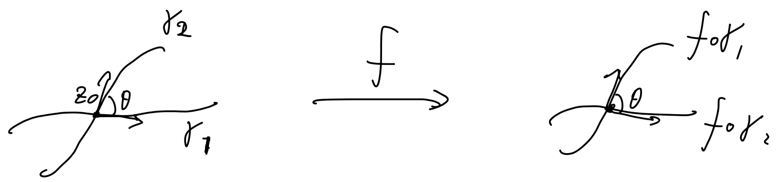
locally each w near w_0 has k preimages near z_0.



$f(z) - w \sim \boxed{(\overset{w_0}{\overset{\|}{f(z_0)}} + (z-z_0)^k \cdot C_k) - w}$ $\quad z - z_0 = \left(\frac{w - w_0}{C_k}\right)^{\frac{1}{k}}$

$|\text{Difference}| = |g(z)| = O(\varepsilon^{k+1}) < \overset{\ll \varepsilon^k}{-|w_0 - w|} + C_k \underset{\varepsilon^k}{\underset{\|}{|z-z_0|^k}}$

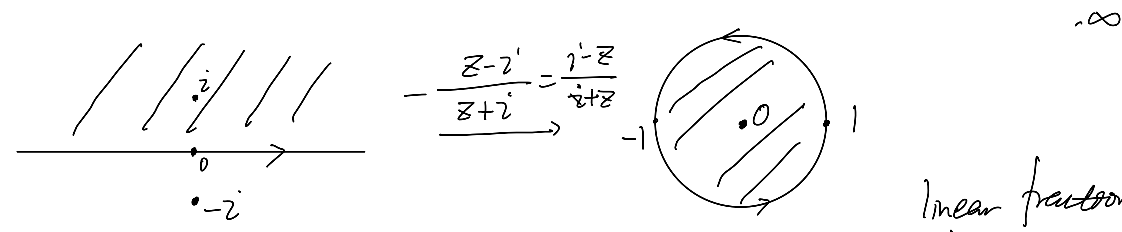
on $|z - z_0| = \varepsilon$



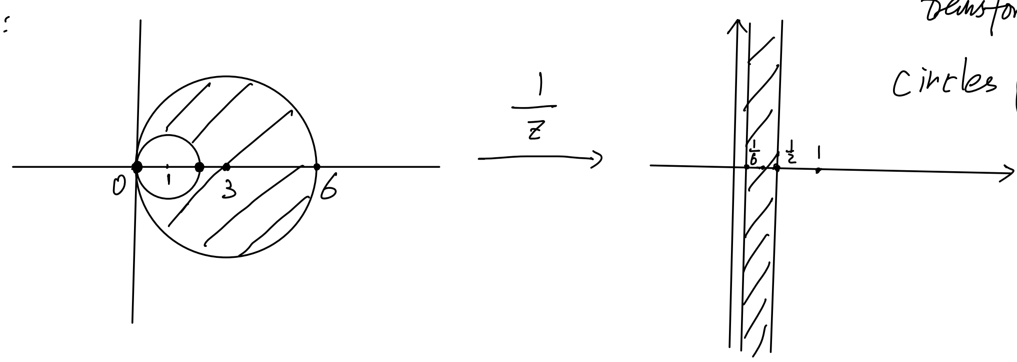
$\angle\, \gamma_1' \, \gamma_2'$ $\qquad$ $(f\circ\gamma_i)' = f'(z_0)\cdot\gamma_i'$

Riemman Mapping Thm: Any simply connected domain $\neq \mathbb{C}$ is conformal to the unit disk.

Ex:



Ex:



linear fractional transformation preserves circles (lines = circle passing through infinity)

$$\left(z_1 - \frac{1}{6}\right)\cdot\frac{\pi i}{\frac{1}{3}} = \underset{z_2}{\underset{\shortparallel}{3\pi i\left(z_1 - \frac{1}{6}\right)}}$$