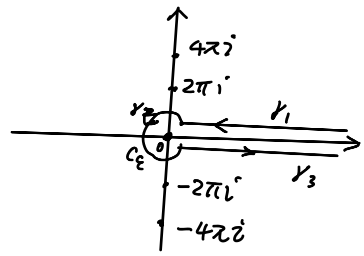
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ locally converges uniformly on $\underset{\sigma}{\underset{\shortparallel}{\mathrm{Re}\, s}} > 1$

holomorphic fct.

when $\overset{\sigma}{\overset{\shortparallel}{\mathrm{Re}\, s}} > 1$:

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_{C_\varepsilon} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$ entire



$s = \sigma + it$

$$\int_{C_\varepsilon} \left| \frac{(-z)^{s-1}}{e^z - 1} \right| \le \int_{\varepsilon}^{\infty} \frac{|z|^{\sigma-1}}{e^{\underset{x}{\underset{\shortparallel}{\mathrm{Re}\, z}}} - 1}\, dx$$

$\Gamma(1-s)$: $\underline{\text{has poles at } 1-s = -n,\ s = 1+n,\ n = 0, 1, 2, \dots}$

$$\frac{1}{2\pi i} \int_{C_\varepsilon} \frac{(-z)^{n-1}}{e^z - 1}\, dz = (-1)^{n-1} \underbrace{\mathrm{res}_{z=0} \left(\frac{z^{n-1}}{e^z - 1} \right) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}}$$

$\Rightarrow$ the only of $\zeta(s)$ is at $s = 1$, $\mathrm{res}_{s=1} \zeta(s) = 1$.

$$\Rightarrow \zeta(-n) = (-1)^n \frac{\Gamma(1+n)}{2\pi i} \int_C \frac{z^{-n-1}}{e^z - 1}\, dz$$

$$= (-1)^n\, n!\ \underbrace{\mathrm{res}_{z=0} \left(\frac{z^{-n-1}}{e^z - 1} \right)} = \begin{cases} -\frac{1}{2} & n = 0 \\ \frac{1}{12} & n = 1 \\ \dots & \end{cases}$$

$$\mathrm{res}_{z=0} \frac{z^{-1}}{e^z - 1} = \frac{d}{dz} \left(z^2 \cdot \frac{z^{-1}}{e^z - 1} \right) \Big|_{z=0}$$

$$\frac{1}{z^{n+2}} \left(1 - \frac{z}{2} + \frac{z^2}{12} - \dots \right)$$

$$\frac{\frac{1}{z^{n+1}}}{z + \frac{z^2}{2} + \frac{z^3}{3!}} = \frac{1}{z^{n+2}} \cdot \frac{1}{1 + \frac{z}{2} + \frac{z^2}{6} + \dots} = \frac{1}{z^{n+2}} \Big(\underbrace{1 - \left(\frac{z}{2} + \frac{z^2}{6} + \dots \right) + \overset{\frac{z^2}{4}}{\left(\frac{z}{2} + \frac{z^2}{6} + \dots \right)^2} + \dots}_{-\frac{1}{6} + \frac{1}{4}} \Big)$$

$$\frac{1}{1^0}+\frac{1}{2^0}+\frac{1}{3^0}+\cdots$$

$$\shortparallel$$

$$\zeta(0)=(-1)^0\,0!\cdot\left(-\frac{1}{2}\right)=-\frac{1}{2}=1+1+1+\cdots$$

$$\zeta(-1)=(-1)^1\,1!\cdot\left(-\frac{1}{12}\right)=-\frac{1}{12}=1+2+3+4+\cdots$$

$$\shortparallel$$

$$\frac{1}{1^{-1}}+\frac{1}{2^{-1}}+\frac{1}{3^{-1}}+\cdots$$

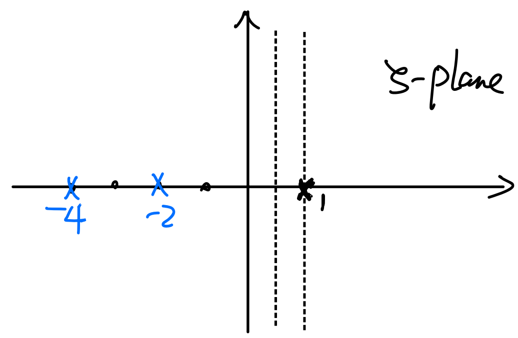
$$\shortparallel$$

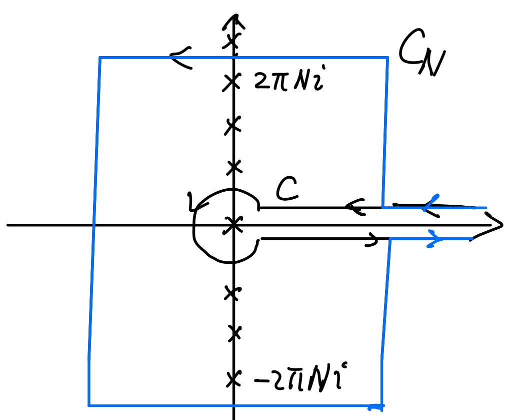
$$1+2+3+\cdots$$

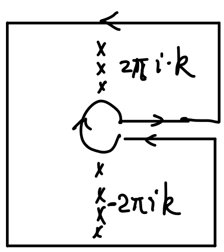
$$\zeta(-2m+1)=(-1)^m\frac{B_m}{2m}$$

$$\zeta(-2m)=0$$

$$\zeta(s)=-\frac{\Gamma(1-s)}{2\pi i}\int_{C_\varepsilon}\frac{(-z)^{s-1}}{e^z-1}\,dz$$







$$\frac{1}{2\pi i}\int_{C_N-C_\varepsilon}\underbrace{\frac{(-z)^{s-1}}{e^z-1}}_{f(z)}\,dz=\sum_{k=1}^{N}\operatorname{res}_{z=2\pi ik}f(z)+\sum_{k=1}^{N}\operatorname{res}_{z=-2\pi ik}f(z).$$

$$\shortparallel$$

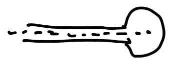
$$\int_{C_N}-\int_{C_\varepsilon}$$

$$f(z) = \frac{(-z)^{s-1}}{e^z - 1}, \qquad (-z)^{s-1} = e^{(s-1)\cdot\log(-z)}$$

$$\operatorname{res}_{z=2\pi i k} f(z) = \lim_{z\to 2\pi i k} (z-2\pi i k)\cdot\frac{(-z)^{s-1}}{e^z - 1} = (-2\pi i k)^{s-1} = e^{(s-1)\cdot\log(-2\pi i k)}$$

$$\log(-2\pi i k) = \left(\log(2\pi k) - i\cdot\frac{\pi}{2}\right)$$

$$e^z - 1 = e^{z-2\pi i k} - 1 = (z-2\pi i k) + \frac{(z-2\pi i k)^2}{2} + \cdots$$



$$\operatorname{res}_{z=-2\pi i k} f(z) = (2\pi i k)^{s-1}$$

$$\frac{1}{2\pi i}\int_{C_N - C_\varepsilon} \frac{(-z)^{s-1}}{e^z - 1}\,dz = \sum_{k=1}^{N}\left[(-2\pi i k)^{s-1} + (2\pi i k)^{s-1}\right]$$

$$= \sum_{k=1}^{N} k^{s-1}\cdot(2\pi)^{s-1}\cdot\left((-i)^{s-1} + i^{s-1}\right)$$

$$(-i)^{s-1} + i^{s-1} = \frac{e^{-i\frac{\pi}{2}(s-1)} + e^{i\cdot\frac{\pi}{2}\cdot(s-1)}}{2}\cdot 2 = \cos\left(\frac{\pi}{2}(s-1)\right)\cdot 2 = \sin\left(\frac{\pi s}{2}\right)\cdot 2$$

$$= 2\cdot(2\pi)^{s-1}\left(\sum_{k=1}^{N}\frac{1}{k^{1-s}}\right)\cdot\sin\frac{\pi s}{2}$$

$$\left|\frac{(-z)^{s-1}}{e^z-1}\right| \le C N^{\operatorname{Re}\sigma - 1}$$

Let $N\to+\infty$, if $\operatorname{Re} s = \sigma < 0$,

$$-\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{(-z)^{s-1}}{e^z-1}\,dz = 2\cdot(2\pi)^{s-1}\cdot\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\sin\left(\frac{\pi s}{2}\right)$$

$$-\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{(-z)^{s-1}}{e^z-1}\,dz = \frac{\zeta(s)}{\Gamma(1-s)}, \qquad \sum_{k=1}^{\infty}\frac{1}{k^{1-s}} = \zeta(1-s)$$

$$\Rightarrow \boxed{\zeta(s) = 2^s\cdot\pi^{s-1}\cdot\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\cdot\zeta(1-s)} \quad \text{for any } s\in\mathbb{C}$$

$\xi(s) = \frac{1}{2} s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ entire fct.

$\frac{s}{2} = -m$. $s = -2m$

$m \geq 0$

$\Rightarrow \boxed{\xi(1-s) = \xi(s)}$ for all $s \in \mathbb{C}$.

zeros of ξ = non trivial zeros of ζ

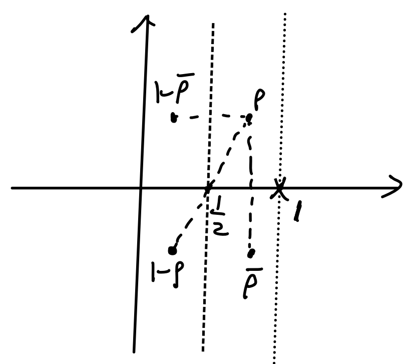
Riemann Hypothesis : $\{\text{zeros of } \xi\} \subset \{\operatorname{Re} s = \frac{1}{2}\}$

• Fact: $|\xi(s)| \leq A \cdot e^{B|s| \cdot \log|s|}$ $\forall s \in \mathbb{C}$

order 1

Hadamard Factorization Thm $\Longrightarrow$ $\xi(s) = e^{As+B} \prod_{\rho \in Z} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} = \prod_{\rho \in Z} \left(1 - \frac{s}{\rho}\right)$.

$\overline{\xi(s)} = \xi(\overline{s})$



• Thm: $\underline{\zeta(s) \text{ has no zeros on } \operatorname{Re} s = 1}$. (Hadamard, de la Vallée Poussin 1896.)

$\Updownarrow$ Riemann, Chebyshev

$\#\left\{\begin{matrix}\text{prime} \\ \text{numbers} \\ \leq x\end{matrix}\right\} = \pi(x) \sim \frac{x}{\log x}$

$$\zeta(s) = \prod_{\substack{p \text{ prime} \\ \text{numbers}}} (1-p^{-s})^{-1} \quad \text{Euler product formula.}$$

$$\underbrace{\log \zeta(s)} = \sum_p \underbrace{\log\left(\frac{1}{1-p^{-s}}\right)}_{\substack{\| \\ p^{-s} + \frac{p^{-2s}}{2} + \cdots}} = \sum_{p,m} \frac{p^{-ms}}{m} \qquad \overset{\sigma}{\overset{\|}{\mathrm{Re}\, s}} > 1$$

$$= \underbrace{\sum_{n=1}^{\infty} c_n \cdot n^{-s}} \qquad c_n \geq 0$$

$c_n \neq 0$ if $n = p^m$ for some $m \in \mathbb{Z}_{>0}$

Lem: If $\sigma > 1$, then

$$\underbrace{\log \left| \zeta^3(\sigma)\, \zeta^4(\sigma + it) \cdot \zeta(\underset{\sigma + i(2t)}{\sigma + 2it}) \right| \geq 0}.$$

Pf: $\log|\zeta(s)| = \mathrm{Re}(\log \zeta(s)) = \mathrm{Re}\left(\sum_{n=1}^{\infty} c_n \underset{n^{-\sigma - it} = n^{-\sigma}(\underline{\cos(t)} + i \cdot \sin(t))}{n^{-s}}\right) = \sum_{n=1}^{\infty} c_n \cdot n^{-\sigma} \cdot \cos t)$

$$\Rightarrow 3 \log|\zeta(\sigma)| + 4 \cdot \log|\zeta(\sigma + it)| + \log|\zeta(\sigma + i \cdot 2t)|$$

$$= \sum_{n=1}^{\infty} c_n \cdot n^{-\sigma} \cdot \underbrace{(3 + 4 \cdot \cos(t) + \cos(2t))}_{2(1+\cos\theta)^2}$$

$\blacksquare$

Proof of Thm by contradiction.

Suppose $\zeta(\underbrace{1 + i \cdot t_0}_{\rho_0}) = 0$. Then $\zeta(s) = (\underset{\sigma + it_0}{s} - \underset{1 + it_0}{\rho_0}) \cdot h(s)$

$$\underline{|\zeta(\sigma + it_0)|^4 \leq C \cdot (\sigma - 1)^4}$$

$\boxed{1 \text{ is a pole of order } 1 \text{ of } \zeta}$ $\zeta(s) = (s-1)^{-1} \cdot h_1(s)$

$$\Rightarrow \underline{|\zeta(\sigma)|^3 \leq (\sigma - 1)^{-3} \cdot C}.$$

$\underline{|\zeta(\sigma+(2t_0)i)| \le C}$

$$\Rightarrow \quad |\zeta(\sigma)|^3 \cdot |\zeta(\sigma+it_0)|^4 \cdot |\zeta(\sigma+(2t_0)\cdot i)|$$

$$\le (\sigma-1)^{-3}\cdot(\sigma-1)^{4}\cdot C = C\cdot(\sigma-1) \xrightarrow{\sigma\to 1} 0 .$$

$$\Rightarrow \quad \log(\qquad) \to \log 0 \to -\infty . \quad \to\leftarrow$$

$\Rightarrow$ Thm $\Leftrightarrow$ prime number theorem. ∎