$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t}\, dt \qquad \operatorname{Re} s > 0.$$

$\rightsquigarrow$ extends to a meromorphic function on $\mathbb{C}$

with poles at $-n$ with residue $(-1)^n \frac{1}{n!}$.

of order 1 $\{0, -1, -2, \cdots\}$

$$\Gamma(s+1) = s \cdot \Gamma(s) \Rightarrow \Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)}$$

$$= \cdots = \frac{\Gamma(s+n+1)}{s\cdot(s+1)\cdots(s+n)} \qquad \operatorname{Re} s > -n-1$$

$$\Gamma(n+1) = n! = n\cdot(n-1)\cdot(n-2)\cdots 1.$$

$$\underbrace{\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{\sin \pi s}} \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Rightarrow \frac{1}{\Gamma(s)} = \Gamma(1-s) \cdot \frac{\sin \pi s}{\pi}$$ zeros of order 1 at $s = \pm n,\ n = 0, 1, \cdots$

$(s-n)\cdot g(s)$

poles of order 1

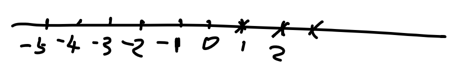
$\frac{1}{s-n}\cdot h(s)$

at $1-s = -n \Leftrightarrow s = 1+n$, $n = 0, 1, 2, \cdots$

$1, 2, 3, \cdots$

$\frac{1}{\Gamma(s)}$ has zeros exactly at $s = 0, -1, -2, \cdots$

$$\| \qquad e^{P(z)} s \cdot \prod_{n=1}^{\infty} \left(1 - \frac{s}{-n}\right) \cdot e^{\frac{s}{-n} + \cdots + \frac{1}{k}\left(\frac{s}{-n}\right)^k}$$



$k \le \rho < k+1$ $\quad \rho$ is the order of growth of $\frac{1}{\Gamma(s)} = f(s)$

$|s|^{1+\varepsilon}$ $\forall \varepsilon > 0$ $\qquad f(s) \le A \cdot e^{B\cdot|z|^{\rho}}$

Thm: $\left|\frac{1}{\Gamma(s)}\right| \le c_1 \cdot e^{c_2 |s| \cdot \log|s|} \Rightarrow \rho = 1+\varepsilon \quad \forall \varepsilon > 0$

$$\Rightarrow \quad \frac{1}{\Gamma(s)} = \left(s\prod_{n=1}^{\infty}\left(1-\frac{s}{-n}\right)\cdot e^{\frac{s}{-n}}\right) e^{As+B}$$

Hadamard Factorization Thm

$$= e^{As+B}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right) e^{-\frac{s}{n}}$$

$$\Rightarrow \quad \frac{1}{\Gamma(s)\cdot s} = e^{As+B}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)\cdot e^{-\frac{s}{n}}$$

$\downarrow s\to 0$ $\qquad \| s=0$

$\Gamma(s) = \frac{1}{s}(-1)^0 + g(s)$

$$1 = e^{B}\cdot 1 \Rightarrow e^{B} = 1.$$

$s\cdot\Gamma(s) = 1 + s\cdot g(s)$

$\downarrow$

1

$$\frac{1}{\Gamma(s)} = e^{As}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

$$\xRightarrow{s=1} \quad 1 = e^{A}\cdot\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = e^{A}\cdot e^{\left(\sum_{n=1}^{\infty}\left(\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)\right)}$$

$-\gamma$

$\Gamma(1) = 0! = 1$

$$\sum_{n=1}^{N}\log\left(1+\frac{1}{n}\right) = \sum_{n=1}^{N}\log\frac{n+1}{n} = \log\prod_{n=1}^{N}\frac{n+1}{n} = \log(N+1).$$

$$\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{N+1}{N}$$

$$\log(N+1) - \sum_{n=1}^{N}\frac{1}{n}$$

$$\log\frac{N+1}{N} + \boxed{\left(\log N - \sum_{n=1}^{N}\frac{1}{n}\right)}$$

$\downarrow N\to\infty$ $\qquad \downarrow$

$\log 1 = 0$ $\qquad -\gamma$

$$\sum_{n=1}^{N}\frac{1}{n} - \log N \xrightarrow{N\to+\infty} \gamma$$

Euler's constant.

$$\Rightarrow e^{A} = e^{\gamma}$$

$$\Rightarrow \quad \frac{1}{\Gamma(s)} = \underline{e^{\gamma\cdot s}\cdot s\cdot\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)\cdot e^{-\frac{s}{n}}}$$

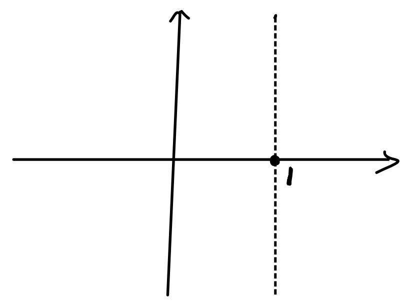
$$\frac{\sin\pi s}{\pi} = s\prod_{n=1}^{\infty}\left(1-\frac{s}{n}\right)e^{\frac{s}{n}}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

· Zeta function. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$s=\sigma+it \quad \left|\frac{1}{n^s}\right| = |n^{-s}| = |n^{-\sigma-it}| = n^{-\sigma}$

$\left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges absolutely $\underset{\text{Re}\,s}{\overset{\shortparallel}{\sigma}} > 1$.

$\Rightarrow$ $\zeta(s)$ is holomorphic on $\{\text{Re}\, s > 1\}$.



$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx$$

$$= \int_0^\infty (n\cdot t)^{s-1} e^{-n\cdot t}\, d(n t)$$

$$= n^s \cdot \int_0^\infty t^{s-1} e^{-nt}\,dt.$$

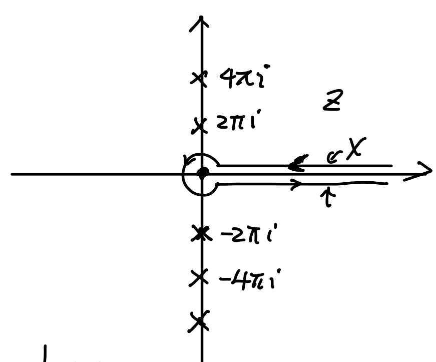
$$\left(\sum_{n=1}^{\infty} n^{-s}\right)\Gamma(s) = \sum_{n=1}^{\infty}\int_0^\infty t^{s-1} e^{-nt}\,dt \qquad \overset{\sigma}{\shortparallel}\ \text{Re}\,s > 1$$

$$\zeta(s)\cdot\Gamma(s) = \int_0^\infty t^{s-1} \left(\sum_{n=1}^{\infty} e^{-nt}\right) dt = \underline{\int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt.}$$

$$\frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t-1} \qquad \underline{\sigma = \text{Re}\,s > 1.}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt$$

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz$$



$e^z = 1 \Leftrightarrow z = 2\pi i\cdot k$, $k = 0, \pm 1,$

$$\left|\frac{(-z)^{s-1}}{e^{z}-1}\right|, \quad |(-z)^{s-1}| = |z|^{\mathrm{Re}\,s-1}, \quad e^{z}-1 \sim z+o(|z|^2)$$

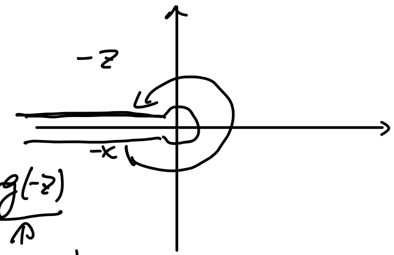
$$\wedge|$$

$$\varepsilon^{\mathrm{Re}\,s-1} \sim |z|^{\mathrm{Re}\,s-1}$$

$$C\cdot\frac{|z|^{\mathrm{Re}\,s-1}}{|z|\sim\varepsilon}\cdot 2\pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

$$(-z)^{s-1} = e^{(s-1)\cdot\log(-z)}$$

$$\log(-z) = \log|-z| + i\cdot\underbrace{\arg(-z)}_{\in[-\pi,\pi)}$$



$$\int_{\infty}^{0}\frac{(-x)^{s-1}}{e^{x}-1}\,dx = -\int_{0}^{\infty}\frac{e^{(s-1)(\log x-\pi i)}}{e^{x}-1}\,dx$$

$$= -\int_{0}^{\infty}\frac{x^{s-1}\cdot e^{-(s-1)\cdot\pi i}}{e^{x}-1}\,dx$$

$$\int_{0}^{\infty}\frac{e^{(s-1)\cdot(\log x+\pi i)}}{e^{x}-1}\,dx = \int_{0}^{\infty}\frac{x^{s-1}}{e^{x}-1}\,dx\cdot e^{(s-1)\pi i}$$

$$\int_{C}\frac{(-z)^{s-1}}{e^{z}-1}\,dz = \underbrace{\left(\int_{0}^{\infty}\frac{x^{s-1}}{e^{x}-1}\,dx\right)}_{=\zeta(s)\cdot\Gamma(s)}\cdot\underbrace{\left(e^{(s-1)\pi i}-e^{-(s-1)\pi i}\right)}_{=2i\cdot\sin((s-1)\pi)}$$

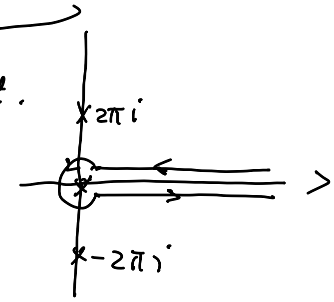
$$\Rightarrow \zeta(s)\cdot\Gamma(s) = \frac{1}{2i\cdot\underbrace{\sin((s-1)\pi)}_{=-\sin(\pi s)}}\int_{C}\frac{(-z)^{s-1}}{e^{z}-1}\,dz.$$

$$\zeta(s) = -\underbrace{\left(\frac{1}{\Gamma(s)}\cdot\frac{1}{\sin(\pi s)}\right)}_{=\frac{\Gamma(1-s)}{\pi}}\cdot\frac{1}{2i}\int_{C}\cdots$$

$$\sin(\theta-\pi) = -\sin\theta.$$

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\cdot\frac{\sin\pi s}{\pi}$$

$$\Rightarrow \quad \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\underbrace{\int_{C_\varepsilon}\frac{(-z)^{s-1}}{e^z-1}\,dz}_{\text{entire fct.}} \qquad \underline{\forall\, 0<\varepsilon<2\pi}$$



$$\left|(-z)^{\sigma+it-1}\right| = |z|^{\sigma-1}$$

$$\int_0^{+\infty}\left|\frac{(-z)^{s-1}}{e^z-1}\right| \le \int_0^{\infty} C\cdot e^{-\frac{x}{2}}\,dx < +\infty$$

$$\frac{1}{2\pi i}\int_C \frac{1}{e^z-1}\,dz = \mathrm{res}_{z=0}\frac{1}{(e^z-1)} = 1.$$

$\zeta(s)$ has a pole of order 1 at $s=1$. $\mathrm{res}_{s=1}\zeta = 1$.

$$\zeta(-n) = -\frac{\overset{n!}{\overbrace{\Gamma(1+n)}}}{2\pi i}\int_C \frac{\overbrace{(-z)^{-n-1}}^{=(-1)^{n+1}\cdot z^{-n-1}}}{e^z-1}\,dz$$

$$= (-1)^n\cdot\frac{n!}{2\pi i}\underbrace{\int_C \frac{z^{-n-1}}{e^z-1}\,dz}_{\substack{\shortparallel \\ \mathrm{res}_{z=0}\left(\frac{z^{-n-1}}{e^z-1}\right)}}$$

$$\boxed{\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1} \quad \leftarrow \text{Bernouli numbers}}$$

$$\Rightarrow \quad \zeta(0) = -\frac{1}{2}, \quad \boxed{\zeta(-2\cdot m) = 0.}$$

$$\zeta(-2m+1) = (-1)^m \frac{B_m}{2m}$$

$$\zeta(-1) = -1 \cdot \frac{\frac{1}{6}}{2} = \boxed{-\frac{1}{12} = 1+2+3+\cdots}$$

$$\|$$

$$\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \cdots$$

$$\|$$

$$1+2+3+\cdots$$

$$\frac{B_1}{2}$$

$$\|$$

$$-\frac{1}{6} + \frac{1}{4} = \frac{1}{12}$$

$$\frac{1}{e^z - 1} = \frac{1}{\left(z + \frac{z^2}{2} + \frac{z^3}{3!}\right)} = \frac{1}{z}\cdot\left(1 - \left(\frac{z}{2} + \frac{z^2}{6} + \cdots\right) + \left(\frac{z}{2} + \frac{z^2}{6} + \cdots\right)^2 \cdots\right)$$

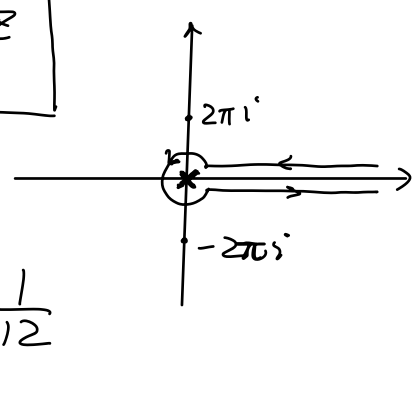
$$\|$$

$$z \cdot \left(1 + \frac{z^1}{2} + \frac{z^2}{6} + \cdots\right)$$

$$\|$$

$$\frac{1}{z} - \frac{1}{2} + \frac{1}{12} z$$

$$\boxed{\zeta(z) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz}$$

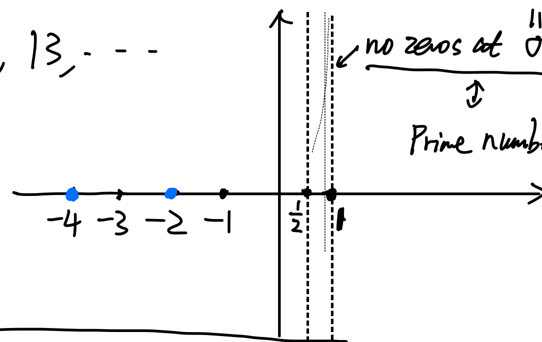


$$\Rightarrow \quad \zeta(-2m) = 0. \qquad \zeta(0) = -\frac{1}{2}$$

$$\zeta(-2m+1) = (-1)^m \frac{B_m}{2m}. \qquad \zeta(-1) = -\frac{1}{12}$$

Prime Numbers: 2, 3, 5, 7, 11, 13, ...

$\pi(x) = \#\{\text{prime numbers} \leq x\}$



no zeros at $\sigma = 1$ ($\sigma \overset{\text{Re } s}{=}$)

$\Updownarrow$

Prime number Thm

Legendre, Gauss: $\boxed{\pi(x) \sim \frac{x}{\log x} \text{ as } x \to +\infty}$

$\Updownarrow$

$\zeta(s)$ has no zeros at $\mathrm{Re}\, s = 1$.

1896: Hadamard.
de la Vallée Poussin.

Euler

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ primes numbers}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\| $$

$$\Updownarrow \qquad \prod_{p \text{ prime numbers}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

$$n = p_1^{k_1} \cdots p_r^{k_r} \qquad \frac{1}{p_1^{k_1 s}} \cdot \frac{1}{p_2^{k_2 s}} \cdots \frac{1}{p_r^{k_r}}$$

$$\zeta(s) = \prod_{\rho \text{ zeros}} \left(1 - \frac{s}{\rho}\right) \cdot e^{P(s)} = \prod_{p \text{ prime numbers}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$\rightsquigarrow$ "Explicit formulae" for $\pi(x)$ (by Riemann)