Hadamard Factorization Thm.

$\forall \varepsilon>0, \exists A, B>0$ s.t. $|f(z)| \le A \cdot e^{B \cdot |z|^{\rho+\varepsilon}}$.

f is an entire fct. of growth order ρ.

zeros of $f = \{0, a_1, a_2, \cdots\}$

Then $f = e^{P(z)} \underbrace{z^m \cdot \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)} = \left(1-\frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \frac{1}{2}\frac{z}{a_n} + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k}$

where k is an integer s.t. $k \le \rho < k+1$

$P(z)$ is a polynomial of degree $\le k$.

Ex: $\underline{f \text{ has no zeros on } \mathbb{C}} \Rightarrow f = e^g$

$\underline{f \text{ has order of growth } \rho} \Rightarrow f = e^{p(z)}$ where $\deg p \le \rho$ (p is polynomial)

$\Downarrow$

growth order of $f = \deg p$

$|f(z)| \le A \cdot e^{B \cdot |z|^{\deg p}}$

Ex: $\sin(\pi z)$ order of growth 1, zeros $= \{0, \pm 1, \pm 2, \cdots\}$

$$\Rightarrow \sin(\pi z) = e^{az+b} \cdot z \cdot \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right) \cdot e^{\frac{z}{n}} \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) \cdot e^{-\frac{z}{n}}$$

$$= e^{az+b} \cdot z \cdot \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \pi \cdot e^{az} \cdot z \cdot \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

$$\pi = \lim_{z \to 0} \frac{\sin(\pi z)}{z} = e^b$$

$$\Rightarrow \frac{\sin(\pi z)'}{\sin(\pi z)} = \underset{\substack{\uparrow \\ \text{odd}}}{\pi \cdot \cot(\pi z)} = a + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n}}{1-\frac{z^2}{n^2}} = \underset{\substack{\Downarrow \\ 0}}{a} + \underbrace{\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}}_{\text{odd}}$$

$\Rightarrow a = 0$.

$$\sin(\pi z) = \pi \cdot z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

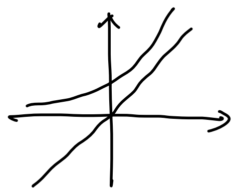
Ex: $|e^z - z| \le |e^z| + |z| \le A \cdot e^{B|z|}$ order of growth $= 1 = k$

$|e^z| = e^{\operatorname{Re} z}$

$$\Rightarrow \quad e^z - z = b \cdot e^{az} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n}}$$

$e^x - x > 0$

$e^x = 1 + x + \frac{x^2}{2}$



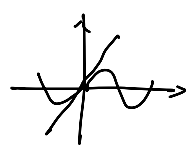
$e^{iy} - i \cdot y$

$= (\cos y + i \cdot \sin y) - i y$

$= \cos y + i(\sin y - y).$

$$e^z - z = \left(1 + z + \frac{z^2}{2!} + \cdots\right) - z$$

$$= 1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$



$e^z - z = b \cdot e^{a \cdot z} \cdot P(z)$. $\Rightarrow a = 1 \Rightarrow (1 - b \cdot P(z)) e^z = z \quad \forall z$.

can't be true.

f has growth order $\le \rho$.

Thm: (i) $n(r) = \#\{\text{zeros of } f \text{ inside } D_r = \{|z| < r\}\}$

$n(r) < C \cdot r^\rho$ for r large.

Pf: $\log|f(0)| = -\sum_{k=1}^{N} \log \frac{R}{|z_k|} + \int_0^{2\pi} \log|f(Re^{i\theta})| \, d\theta$ Jensen's formula

$\{f = 0\} \cap D_R(0) = \{z_1, \ldots, z_N\}$

$\sum_{k=1}^{N} \log \frac{R}{|z_k|} = \int_0^R n(r) \frac{dr}{r}$

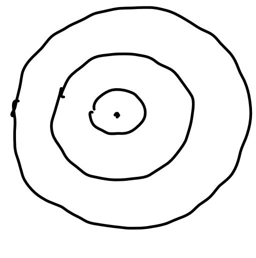
$$\int_r^{2r} n(t)\,\frac{dt}{t} \geq n(r)\cdot\int_r^{2r}\frac{dt}{t} = n(r)\cdot\log 2$$

$$\parallel$$

$$-\log|f(0)| + \int_0^{2\pi} \underbrace{\log|f(2r\cdot e^{i\theta})|}_{\leq\, C\cdot r^{\rho}}\, d\theta \rightsquigarrow \underline{n(r)\leq C'\cdot r^{\rho}}$$

for r large.

(ii) $\boxed{\forall s>\rho,\quad \sum_{n=1}^{\infty}\frac{1}{|z_n|^s} \text{ converges}}$



$$\sum_{|z_n|\geq 1}|z_n|^{-s} = \sum_{j=0}^{\infty}\underbrace{\sum_{2^j\leq|z_n|\leq 2^{j+1}}}_{\leq\, n(2^{j+1})}\underbrace{\frac{1}{|z_n|^{+s}}}_{\leq\, \frac{1}{(2^j)^{+s}}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-j\cdot s}\cdot\underbrace{n(2^{j+1})}_{\leq\, C\cdot 2^{(j+1)\cdot\rho}} \leq C\cdot\sum_{j=0}^{\infty}2^{-j\cdot s}\cdot 2^{(j+1)\rho}$$

$$\leq C\cdot\sum_{j=0}^{\infty}2^{(\rho-s)\cdot j} = \sum_j\left(2^{-(s-\rho)}\right)^j$$

converges if $\underset{k+1}{\overset{\parallel}{s}} > \rho$

$$\Rightarrow \quad z^m\cdot\prod_{n=1}^{\infty}\left(1-\frac{z}{\underset{a_n}{\overset{\parallel}{z_n}}}\right)e^{\frac{z}{a_n}+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k}$$

converges when $k\leq\rho<k+1$

$$\parallel$$

$$z^m\cdot e^{\sum_{n=1}^{\infty}\underbrace{\left(\log\left(1-\frac{z}{a_n}\right)+\frac{z}{a_n}+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k\right)}_{r_n}}$$

$$r_n \leq C\frac{1}{k+1}\cdot\left|\frac{z}{a_n}\right|^{k+1}.\quad \underline{\sum_n|r_n|} \leq C\cdot|z|^{k+1}\cdot\underline{\sum_n\frac{1}{|a_n|^{k+1}}}$$

$$\Rightarrow \quad f = e^{g}\cdot z^m\cdot\prod_{n=1}^{\infty}E_k\left(\frac{z}{a_n}\right),\quad e^{g} = \frac{\overset{\leq\, A\cdot e^{B|z|^s}}{f}}{z^m\cdot\prod_{n=1}^{\infty}E_k\left(\frac{z}{a_n}\right)} \ngeq e^{-c\cdot|z|^s}$$

- $|e^g| = e^{Re(g)}$ satisfies for any $s > \rho$ = order of growth of f

  $Re(g) \leq C \cdot |z|^s$

  $\Rightarrow$ g must be a polynomial.

($f(z)$ entire and $|f(z)| \leq C \cdot |z|^k \Rightarrow f(z)$ is a polynomial of degree $\leq k$.)

$|Re f(z)| \leq C \cdot |z|^k$ $\Rightarrow$

- $\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C \cdot |z|^s}$ for a sequence of radius $|z| = r_m \to +\infty$

$\Rightarrow$ $|Re\, g(z)| \leq C \cdot |z|^s$ when $|z| = r_m \to +\infty$

$\Rightarrow$ $g(z)$ is a polynomial of degree at most s.

$\Rightarrow$ Hadamard Thm.

Gamma function.

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \qquad \text{converges if } s > 0,\ s \in \mathbb{R}.$$

$x^{s-1} = e^{(s-1)\cdot \log x}$ converges if $Re\, s > 0$

$|x^{s-1}| = |x^{\sigma-1} \cdot x^{it}| = x^{\sigma-1}$ $\quad s = \sigma + it$

$|\Gamma(s)| \leq \int_0^\infty x^{\sigma-1} e^{-x}\, dx$ converges when $\sigma > 0$

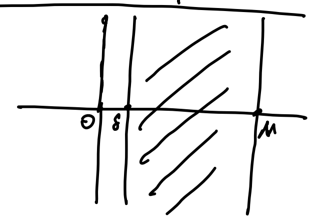
Prop: $\Gamma(s)$ extends to be holomorphic function on $\underbrace{Re(s)>0}_{= U}$.

$$F_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} x^{s-1} \cdot e^{-x}\, dx \qquad \text{holomorphic in } s$$

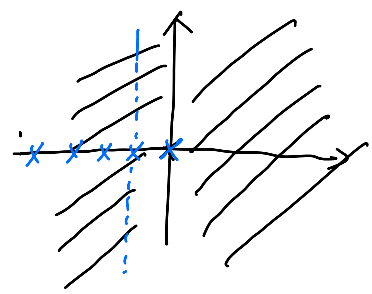
Show $F_\varepsilon \to \Gamma$ uniformly on any $S_{\delta,M}$ set of U.

$$|F_\varepsilon(s) - \Gamma(s)| \le \underbrace{\int_0^\varepsilon x^{\sigma-1} e^{-x}\, dx}_{\le \int_0^\varepsilon x^{\sigma-1}dx = \frac{1}{\sigma}\varepsilon^\sigma \to 0,\ \sigma \ge \delta} + \underbrace{\int_{\frac{1}{\varepsilon}}^{+\infty} x^{\sigma-1} e^{-x}\, dx}_{\le \int_{\frac{1}{\varepsilon}}^{+\infty} x^{M-1}e^{-x}dx \xrightarrow{\varepsilon \to +\infty} 0}$$

$$\Gamma(s) = \int_0^\varepsilon + \int_\varepsilon^{+\infty}$$



$\Rightarrow$ Γ is holomorphic on $\{Re\, s > 0\}$



Lemma: $\boxed{\Gamma(s+1) = s \cdot \Gamma(s). \quad Re(s) > 0}$

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} \cdot e^{-t} dt = \int_0^{+\infty} e^{-t} \frac{dt^s}{s} = \frac{1}{s}\left(e^{-t} \cdot t^s \Big|_0^{+\infty} - \int_0^{+\infty} t^s \cdot e^{-t} (-1) dt \right)$$

$$= \frac{1}{s} \cdot \int_0^{+\infty} t^s \cdot e^{-t} dt = \frac{1}{s} \Gamma(s+1)$$

$$\Rightarrow \Gamma(s) = \frac{\Gamma(s+1)}{s} \qquad Re\, s > -1 \qquad Re(s+1) > 0$$

$$= \frac{\Gamma(s+2)}{s \cdot (s+1)} \qquad Re\, s > -2.$$

$$= \cdots = \frac{\Gamma(s+n)}{s \cdot (s+1) \cdots (s+n)} \qquad Re\, s > -n.$$

$\Rightarrow \Gamma(s)$ defined for any $s \in \mathbb{C}$

$\Gamma(s)$ has poles at $\{0, -1, -2, \ldots\}$

$$\text{res}_{s=-n} \Gamma(s) = \text{res}_{s=-n} \left(\frac{\Gamma(s+n+1)}{s \cdot (s+1) \cdots (s+n)(s+n+1)} \right)$$

$\text{Re}\, s > -n-1.$

$\text{Re}(s+n+1) > 0.$

$$= \frac{\Gamma(1)}{(-n) \cdot (-n+1) \cdots (-2) \cdot (-1) \cdot \underbrace{(-n+n+1)}_{1}}$$

$$= \frac{1}{(-1)^n \cdot n! \cdot 1} = (-1)^n \cdot \frac{1}{n!}$$

$$\boxed{\begin{array}{c} \text{Res}_{z=z_0} \dfrac{f(z)}{(z-z_0)} \\ \shortparallel \\ f(z_0) \end{array}}$$

$\Gamma(n) = (n-1)!$

$\Gamma(1) = 1$

Thm: For all $s \in \mathbb{C}$, $\Gamma(s) \cdot \Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$.

Pf: Both have poles at $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

It is enough to show this is true for $0 < s < 1$.

$$\Gamma(1-s) = \int_0^\infty u^{-s} \cdot e^{-u} du \underset{u = v \cdot t}{\overset{t > 0}{=\!=}} t \int_0^\infty (vt)^{-s} \cdot e^{-v \cdot t} \cdot dv$$

$$\Gamma(s) \cdot \Gamma(1-s) = \int_0^\infty e^{-t} \, t^{s-1} \cdot t \int_0^\infty (vt)^{-s} \cdot e^{-vt} \, dv \qquad -1 < -s < 0.$$

$$= \int_0^\infty \int_0^\infty e^{-t(1+v)} \cdot v^{-s} dt dv = \boxed{\int_0^\infty \frac{v^{-s}}{1+v} dv = \frac{\pi}{\sin \pi(1-s)}}$$

$$= \frac{\pi}{\sin \pi s}.$$

$$\boxed{\Gamma(s)\cdot\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}} \Rightarrow \Gamma\left(\tfrac{1}{2}\right)^2=\frac{\pi}{\sin\frac{\pi}{2}}=\pi \Rightarrow \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$$\frac{\sin(\pi s)}{\pi}=s\cdot\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right)=s\cdot\prod_{n=1}^{\infty}\left(1-\frac{s}{n}\right)\cdot e^{+\frac{s}{n}}\cdot\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)\cdot e^{-\frac{s}{n}}$$

$$=\frac{1}{\Gamma(s)}\cdot\frac{1}{\Gamma(1-s)}.$$

$$\frac{\sin(\pi s)}{s}\cdot\Gamma(1-s)$$

$$\frac{1}{\Gamma(s)} \text{ has zeros at } \{0,-1,-2,\ldots\}$$
