$\{a_n\} \subset \mathbb{C}$ sequence s.t. $\lim_{n\to+\infty} |a_n| = +\infty$.

$\not\ni 0$

Describe holomorphic function f with zeros $\{0\} \cup \{a_n\}$ and no other.

$$z^m \prod_{n=1}^{\infty} \underbrace{\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\frac{z^2}{a_n^2}+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k}} = z^m \prod_n e^{r_n(z)} = z^m \cdot e^{\sum_{n=1}^{\infty} r_n(z)}$$

$$r_n(z) \; \shortparallel$$

$$\left(1-\frac{z}{a_n}\right)\cdot e^{\frac{z}{a_n}+\frac{1}{2}\frac{z^2}{a_n^2}+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k} = e^{\log\left(1-\frac{z}{a_n}\right)+\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k}$$

$$\log(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots$$

$$-\sum_{l=k+1}^{\infty} \frac{1}{l}\left(\frac{z}{a_n}\right)^l$$

$$\shortparallel$$

$$-\frac{z}{a_n}-\frac{1}{2}\left(\frac{z}{a_n}\right)^2-\cdots+\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k = -\frac{1}{k+1}\left(\frac{z}{a_n}\right)^{k+1}-\cdots$$

$$= \left| -\frac{1}{k+1}\left(\frac{z}{a_n}\right)^{k+1}\cdot\left[1+\frac{k+1}{k+2}\left(\frac{z}{a_n}\right)+\frac{k+1}{k+3}\left(\frac{z}{a_n}\right)^2+\cdots\right]\right|$$

$$\leq \frac{1}{k+1}\left|\frac{z}{a_n}\right|^{k+1}\cdot\left(1+\underset{\wedge 1}{\frac{k+1}{k+2}}\left|\frac{z}{a_n}\right|+\frac{k+1}{k+3}\left|\frac{z}{a_n}\right|^2+\cdots\right)$$

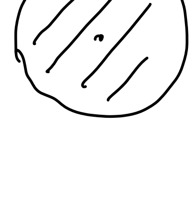
$$1+\left|\frac{z}{a_n}\right|+\left|\frac{z}{a_n}\right|^2+\cdots = \frac{1}{1-\left|\frac{z}{a_n}\right|}$$

$$|r_n(z)| \leq \frac{1}{k+1}\frac{|z|^{k+1}}{|a_n|^{k+1}}\cdot\frac{1}{1-\left|\frac{z}{a_n}\right|}. \qquad \boxed{\sum_{n=1}^{\infty}|r_n(z)| \leq \sum_{n=1}^{\infty}\frac{1}{k+1}\frac{|z|^{k+1}}{|a_n|^{k+1}}\cdot 2}$$

$$z^m \cdot e^{\sum_{n=1}^{\infty} r_n(z)}$$

$|z| \leq K$

$|a_n| \to +\infty$



$\Rightarrow$ If $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}}$ converges, then $\displaystyle\sum_{n=1}^{\infty} |r_n(z)|$ uniformly on any compact set.

$$z^m \cdot e^{\sum_{n=1}^{\infty} r_n(z)} = z^m \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k}.$$

Ex: $\{a_n\} = \mathbb{Z} = \{ \pm 1, \pm 2, \cdots \}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{|a_n|^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \qquad k=1$$

$$\Rightarrow f = z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z}{n}\right) e^{\frac{z}{n}} \cdot \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \quad \text{converges.}$$

$$\frac{\sin(\pi \cdot z)}{f} = e^g \Rightarrow \sin(\pi z) = z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \cdot e^{g}$$

$$+\pi \cdot \cot(\pi z) = \frac{+\cos(\pi z)\pi}{\sin(\pi z)} = \frac{\sin(\pi z)'}{\sin(\pi z)} = \frac{z'}{z} + \sum_{n=1}^{\infty} \frac{\left(1-\frac{z^2}{n^2}\right)'}{1-\frac{z^2}{n^2}} + \frac{(e^g)'}{e^g}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1-\frac{z^2}{n^2}} + g'$$

$$\boxed{\pi \cdot \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}} + g' \Rightarrow g' = 0 \Rightarrow g = \text{const.}$$

$$\sin(\pi z) = z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \cdot C \quad \Rightarrow \boxed{\sin(\pi z) = \pi \cdot z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)}$$

$$\pi = \lim_{z\to 0} \frac{\sin(\pi z)}{z} = \lim_{z \to 0} \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \cdot C = C$$

Ex: $a_n = \{-1, -2, -3, \cdots\}$

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

$$\Rightarrow \boxed{\prod_{n=1}^{\infty} \left(1 - \frac{z}{-n}\right) \cdot e^{\frac{z}{-n}} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \cdot e^{-\frac{z}{n}} = G(z).}$$

$$z \cdot G(z) \cdot G(-z) = z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \cdot e^{-\frac{z}{n}} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} = \frac{\sin(\pi z)}{\pi}$$

$$\left(\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(z) = \frac{1}{z \cdot G(z) \cdot e^{\gamma z}} \right).$$

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zeros of $f(z) = \{0, a_1, a_2, \cdots\}$.

$$\underline{f(z)} = \underline{z^m \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k} \cdot e^{g}}$$

Q: · whether is a k s.t. $\sum \frac{1}{|a_n|^{k+1}}$ converges?

· How to determine k from $f(z)$?

· Can we choose g to be polynomials?

Thm (Hadamard) Suppose f is entire function of $\underline{\text{growth order } \rho_0}$.
Let k be the integer s.t. $k \le \rho_0 < k+1$. If $a_1, a_2, \cdots$ (non-zero) zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k}$$

where P is a polynomial of degree $\le k$, $m = \mathrm{ord}_0 f$.

Def: f has growth order $\le \rho$ if $\exists A, B > 0$ s.t.

$$\underline{|f(z)| \le A \cdot e^{B \cdot |z|^\rho} \text{ for all } z \in \mathbb{C}.}$$

$$\rho_f = \inf \rho : \left(\begin{array}{l} \forall \varepsilon > 0 \ \exists A_\varepsilon, B_\varepsilon > 0 \text{ s.t.} \\ |f(z)| \le A_\varepsilon \cdot e^{B_\varepsilon \cdot |z|^{\rho_0 + \varepsilon}} \text{ for all } z \in \mathbb{C} \end{array} \right)$$

Ex: . $f = P(z)$ polynomial, $\rho_f = 0$

. $$\underset{f}{\underset{\shortparallel}{|e^{z^k}|}} = |e^{\mathrm{Re}(z^k) + i\,\mathrm{Im}(z^k)}| = |e^{\mathrm{Re}(z^k)}| \le e^{|z|^k} \Rightarrow \rho_f = k$$

$P_1(z) \cdot e^{P(z)}$ $\underline{\text{has order of growth } \deg P}$

Ex: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ growth order ≤ 1

$$|\sin(iy)| = \left| \frac{e^{-y} - e^{y}}{2i} \right| = |\mathrm{sh}(y)| \le \underset{\underset{e^{(1-\varepsilon)y}}{\not\le}}{e^{y}} \Rightarrow \text{growth order} = 1$$

$\sin(\pi z)$ has growth order 1

Hadamard $\Longrightarrow$ $$\sin(\pi z) = e^{P(z)} \cdot z^1 \cdot \prod_{n=1}^{\infty} \left(1-\frac{z}{n}\right) e^{+\frac{z}{n}} \cdot \prod_{n=1}^{\infty} \left(1-\frac{z}{-n}\right) e^{-\frac{z}{n}}$$

$$= e^{P(z)} \cdot z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)$$

$\overset{az+b}{\overset{\parallel}{P}}$ is a polynomial of degree ≤ 1.

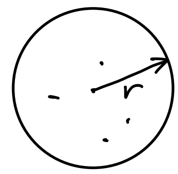
$$\sin(\pi z) = c \cdot e^{az} \cdot z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right). \Longrightarrow c = \pi$$

$$\pi \underbrace{\cot(\pi z)}_{\substack{\uparrow \\ \text{odd}}} = a + \underbrace{\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}}_{\text{odd}} \Longrightarrow a = 0$$

$$\Rightarrow \boxed{\begin{aligned} \sin(\pi z) &= \pi \cdot z \cdot \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \\ \pi \cdot \cot(\pi z) &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \end{aligned}}$$

· f entire function. zeros of $f = \{\overset{0,}{a_1}, a_2, \dots\}$.

$n(r) = \#\{\text{zeros of } f \text{ inside } D_r = \{|z|<r\}\}$
with multiplicities



Thm: If f has growth order $\le \rho$

Then (i) $n(r) \le C\cdot r^{\rho}$ for some $C>0$ and all sufficiently large r.

(ii) If $z_1, z_2, \dots$ zeros of f with $z_n \neq 0$, then for all $s>\rho$

we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|z_n|^s} < \infty$. $\left(\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < +\infty \right)$

Ex: $\sin(\pi z)$ growth order $=1$.

zeros $= \{0, \pm 1, \pm 2, \dots\}$ $\displaystyle\sum_{n\neq 0} \frac{1}{|n|^s}$ converges iff $s>1$

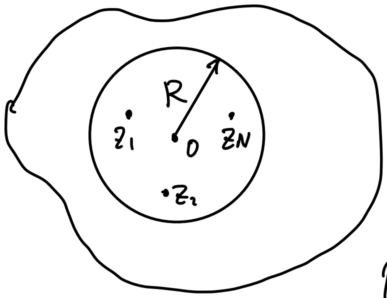
$$k \le \rho_f < k+1 \Rightarrow \overset{\forall \varepsilon>0}{\rho_f+\varepsilon} < s < k+1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < +\infty$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < +\infty$$

$f(0)=0$. $\dfrac{f(z)}{z^m} = g(z)$ $\rho_f = \rho_g$.

- Jensen's formula : f holomorphic in Ω open set.



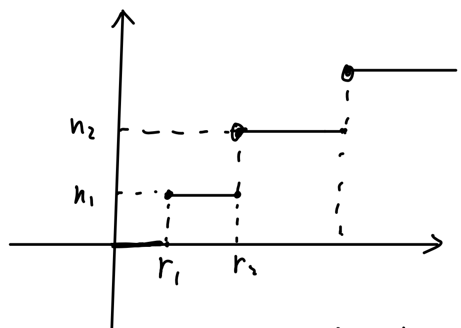
$\{z_1, \cdots, z_N\} \subset D_R(0)$ $f(z_i)=0$, $f(0)\neq 0$.

f is not zero on $C_R = \{|z|=R\}$

Then

$$\log|f(0)| = \sum_{k=1}^{N} \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta.$$

$n(r) = \{\#\text{ zeros of } f \text{ in } D_r\}$.



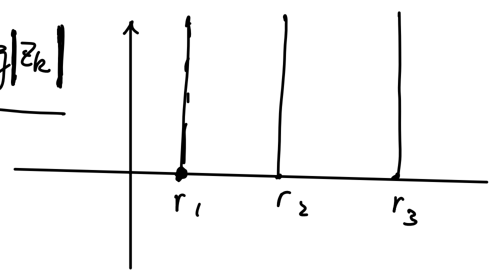
$$\sum_{k=1}^{N} \log \frac{R}{|z_k|} = \int_0^R n(r)\frac{dr}{r}$$

$$= \int_0^R n(r)\, d\log r$$

$$= n(r)\cdot \log r\Big|_0^R - \int_0^R \log r \cdot \underset{\substack{\| \\ n'(r)\,dr}}{d\,n(r)}$$

$$= \underset{\substack{\| \\ N}}{n(R)} \cdot \log R - \sum_k \log|z_k|$$

$\downarrow n'(r)$



$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| - \log|f(0)|$$

$$\int \delta_{r_i}(x)\cdot f(x) = f(r_i)$$

$$R=2r:\quad \int_0^{2r} n(x)\frac{d\theta}{x} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\underset{R}{\underset{||}{2r}}\cdot e^{i\theta})\right| - \log f(0)$$

$$\vee\!|$$

$$\int_r^{2r} n(x)\frac{dx}{x} \geq n(r)\cdot\int_r^{2r}\frac{dx}{x} = n(r)\cdot\log\frac{2r}{r} = n(r)\cdot\log 2$$

$$n(r)\cdot\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi}\underline{\log\left|f(\overset{R}{\overset{||}{2r}}\cdot e^{i\theta})\right|}\,d\theta - \log f(0)$$

$$\wedge\!|$$

$$\log\left(A\cdot e^{B\cdot R^\rho}\right)$$

$$||$$

$$\int_0^{2\pi}\left(\underline{\log A + B} + R^\rho\right)\underline{d\theta}$$

$$\leq C\cdot \underset{2r}{\underset{||}{R^\rho}} = C' r^\rho$$